# The Zonal Mean Climate of the Troposphere

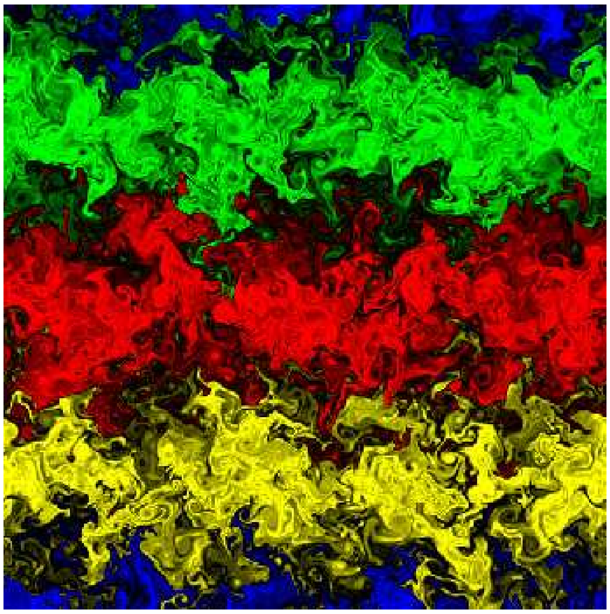
Isaac M. Held

*http://www.gfdl.noaa.gov/$\sim$ ih*

## Goals of my research:

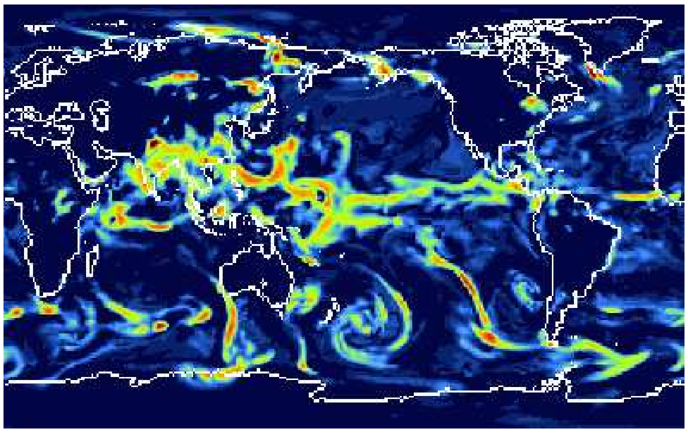
- Develop a *qualitative* understanding of the climate $\Leftarrow$ (these lectures)
- Build bridges across the gap between *idealized models/theory* and *comprehensive climate models*

Mixing of passive tracer in 2D turbulence



**In what sense is this problem $\Uparrow$ simpler than this one $\Downarrow$?**

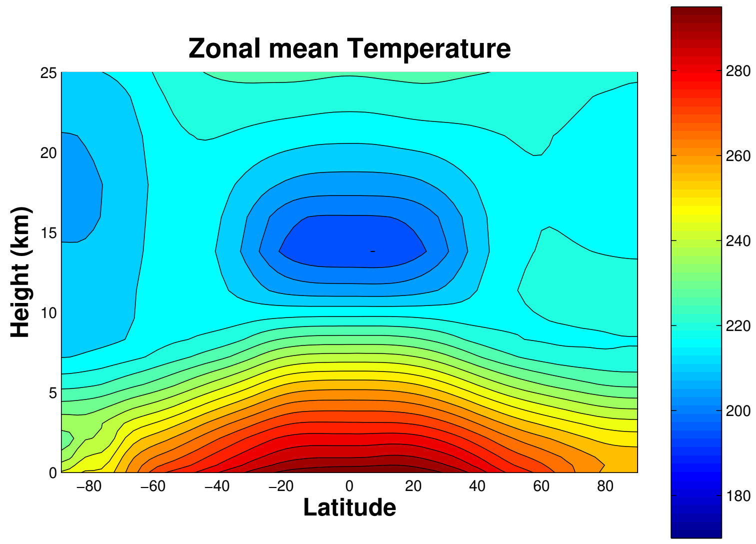
Precipitation in GCM with 1° resolution



# Outline:

- Horizontal angular momentum transport
- Vertical angular momentum transport
- The Hadley cell
- Thermal structure of the extratropical troposphere

# Annual Mean Temperature



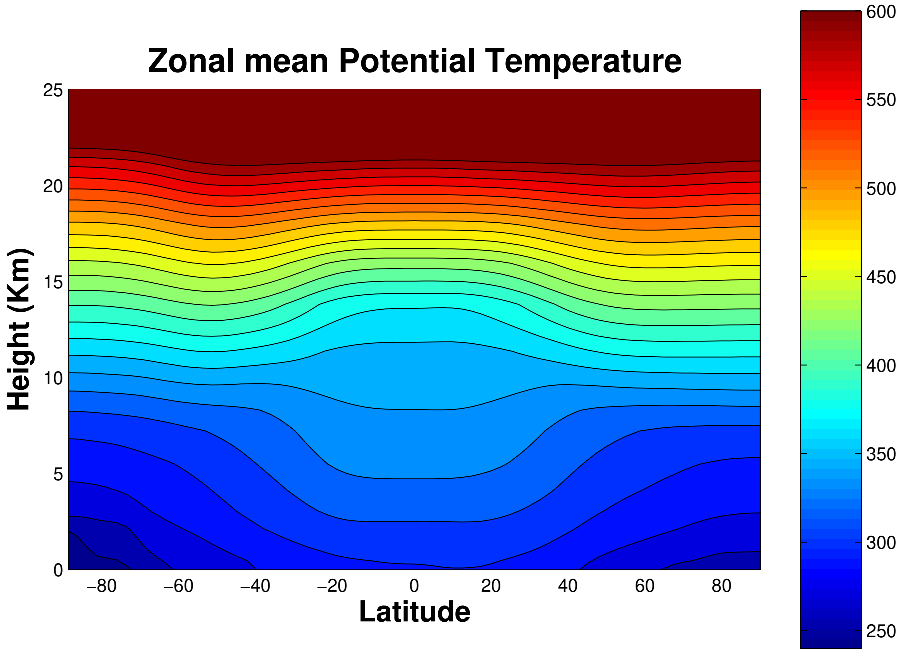

- Why is pole-to-equator temperature difference $\approx 40K$ at the surface
- Why does temperature decrease at $\approx 6.5K/km$ with height in troposphere
- Why does the break between the tropical tropopause at $z \approx 15km$ and the extratropical tropopause at $z \approx 10km$ occur at $\approx 30°$ latitude

# Potential Temperature

entropy of ideal gas $= c_p \, ln(\Theta)$, where

$$\Theta \equiv T \left(\frac{p_*}{p}\right)^{R/c_p} ; \quad N^2 \equiv \frac{g}{\Theta}\frac{d\Theta}{dz}$$

where $N$ is buoyancy frequency.



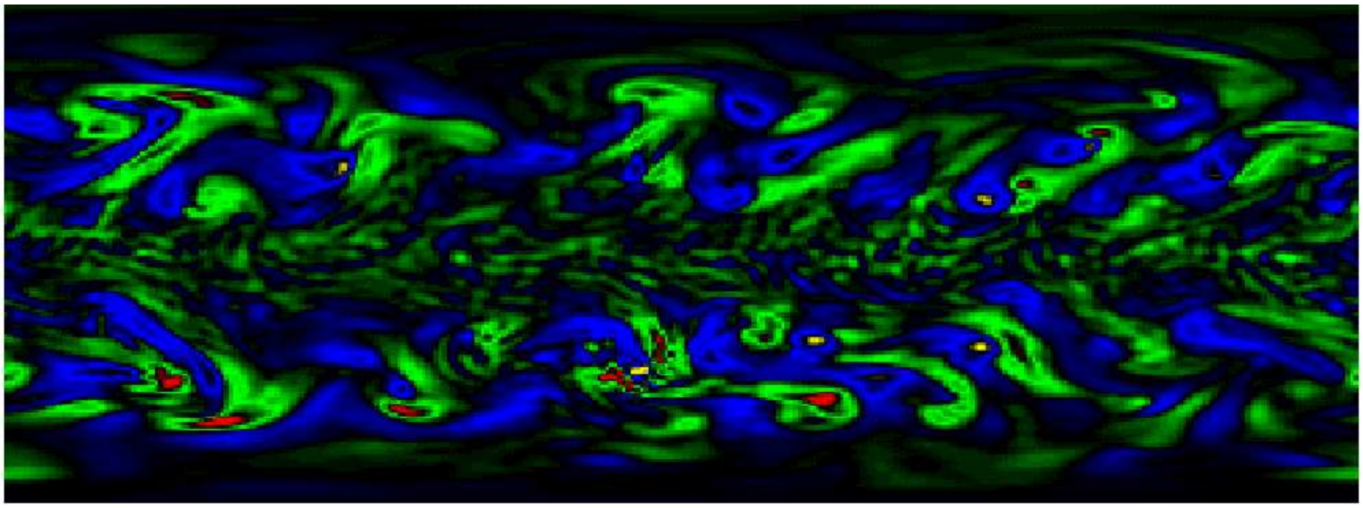

$$\frac{\partial \Theta}{\partial z} \approx 3K/km$$

$$\Rightarrow N \approx 10^{-2} s^{-1}$$

What is the key distinction between Tropics and Extratropics?

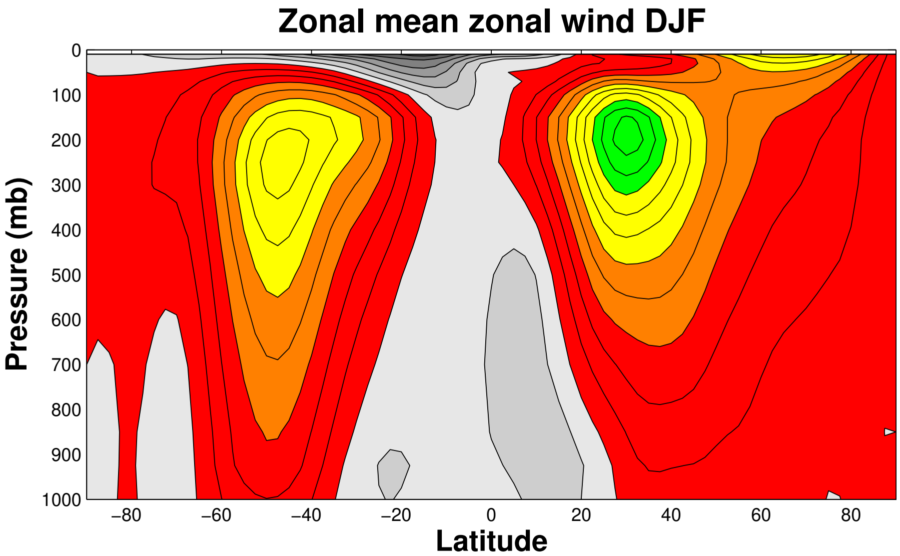
- Extratropics
  - baroclinically unstable, wavelike
  - strong inertial stability;
- Tropics
  - convective
  - weak inertial stability

snapshot of mid-tropospheric vertical motion in an idealized *dry* model:



Distinction between tropics and extratropics exists without water!

# Zonal Winds (DJF)




Hydrostatic plus geostrophic $\rightarrow$ thermal wind relation:

$$\frac{\partial p}{\partial z} = -g\rho; \quad \frac{\partial \Phi}{\partial p} = -\frac{RT}{p}; \quad \Phi \equiv gz$$

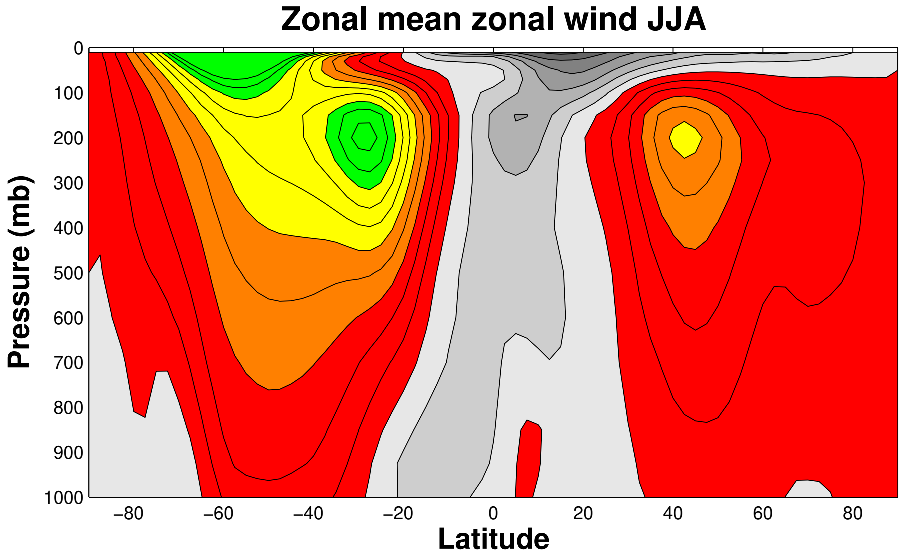
and

$$fU = -\frac{\partial \Phi}{\partial y}$$

$$\Rightarrow f\frac{\partial U}{\partial z} = -g\frac{\partial T}{\partial y}$$

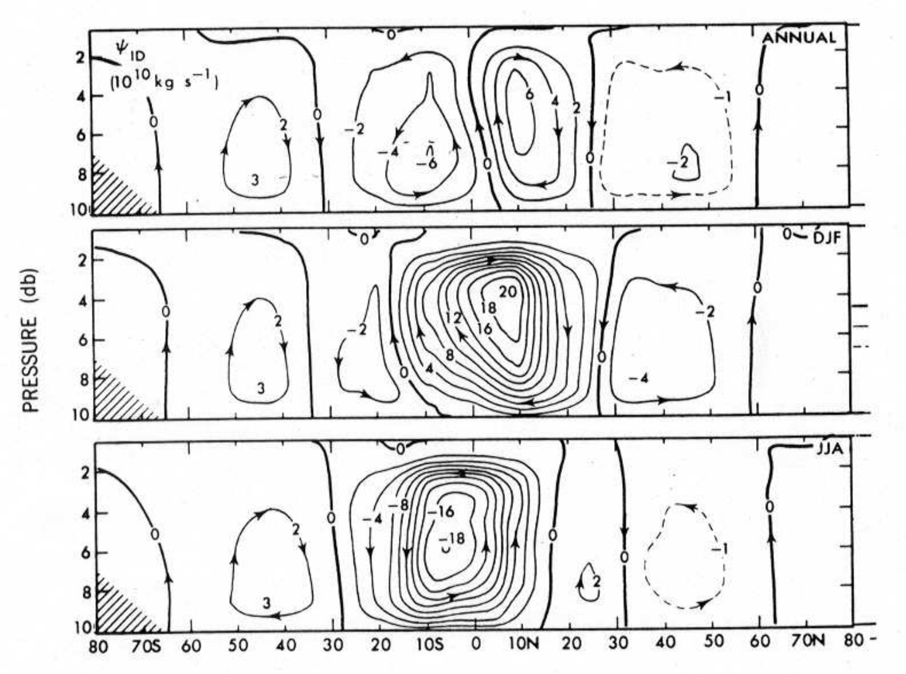
- Westerlies ($U > 0$) throughout the troposphere with strong wintertime subtropical jet
- Easterlies ($U < 0$) at surface equatorward of 30°

# Zonal Winds (JJA)




- Midlatitude westerlies centered poleward of subtropical jets, especially in winter
- In Southern hemisphere winter , surface winds are stronger and westerly maximum over surface westerlies more pronounced, creating hint of double jet

# "Three-cell" overturning circulation



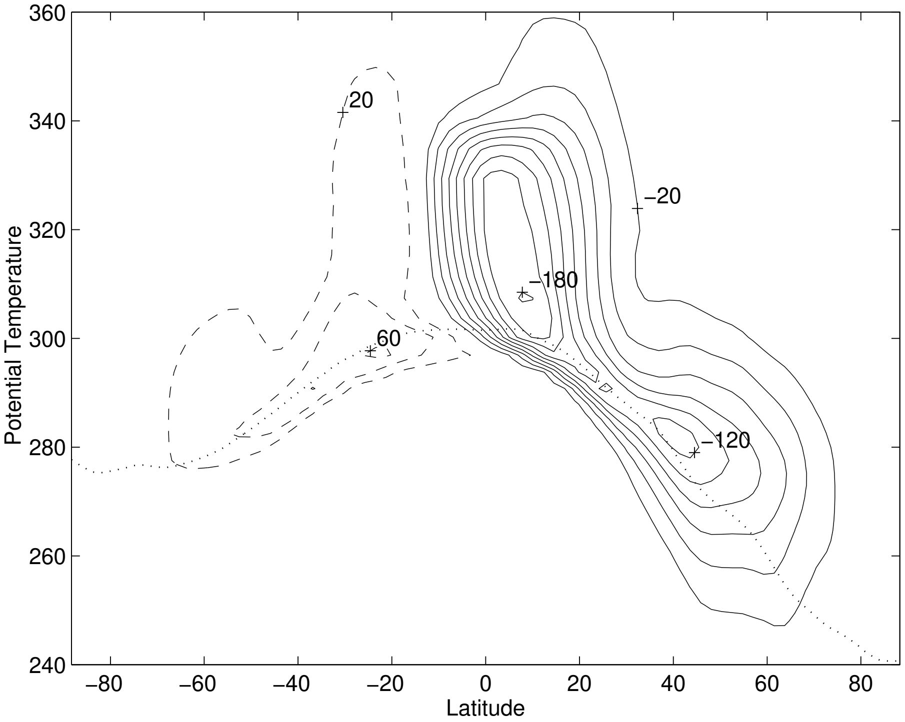

From Peixoto and Oort
Contour interval is 20 *Sverdrups* (Sv $= 10^9\, Kg/s$)

- annual mean tropical structure a residual from larger cross-equatorial soltitial cells
- At low levels, $fV \propto U$

$$\frac{\partial U}{\partial t} = 0 \approx fV + \tau \approx fV - \kappa U \Rightarrow fV \approx \kappa U$$

Picture depends on how one averages:

# Overturning isentropic mass flux circulation

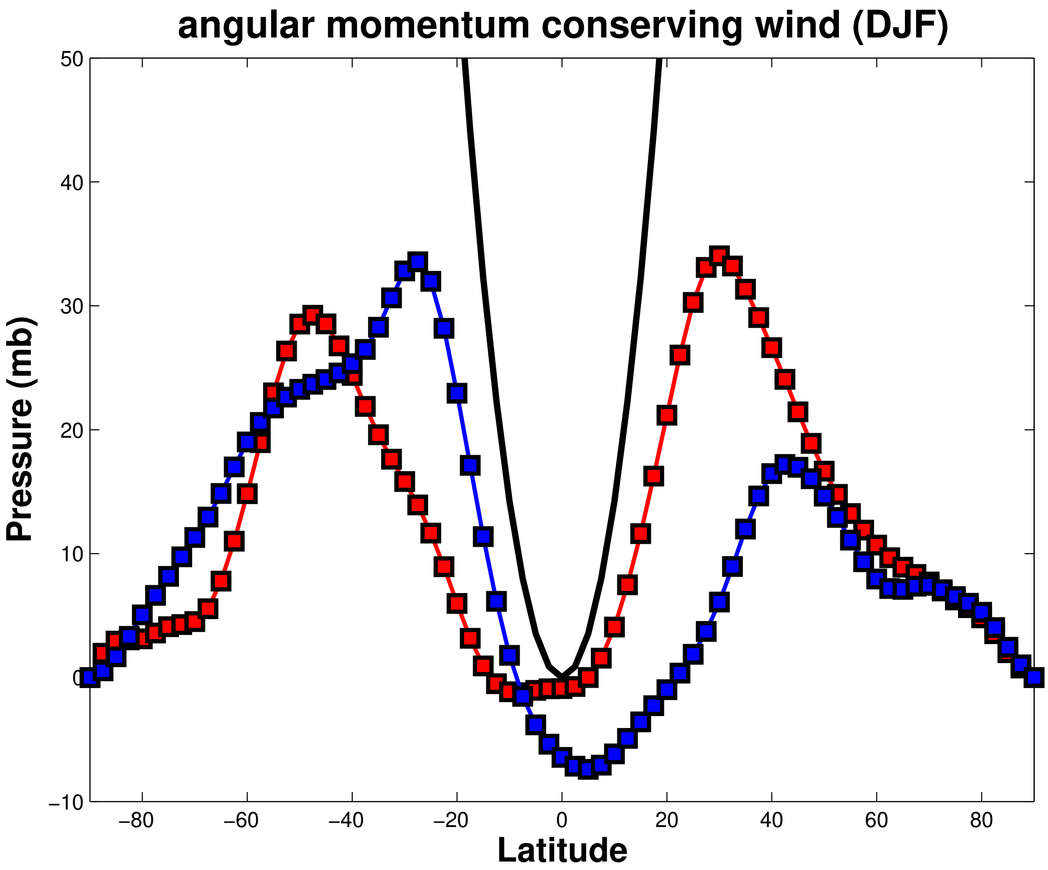



- Vertical circulation is flux across isentropes.
- Horizontal flux is mass flux within infinitesimal layer between two isentropes

$$\overline{v\delta m} = V\overline{\delta m} + \overline{v'(\delta m)'}$$

- dotted line is median surface temperature.

# Angular momentum conserving wind and 200mb zonal winds



Angular momentum per unit mass,

$$M = (U + \Omega a \cos\theta) a \cos\theta$$

$U = 0$ at $\theta = 0$, $\Rightarrow M(0) = \Omega a^2$.

Conserving $M$ as one moves poleward

$$\Rightarrow M = \Omega a^2 = (U + \Omega a \cos\theta) a \cos\theta$$

$$\Rightarrow U(\theta) = \frac{\Omega a \sin^2(\theta)}{\cos(\theta)} \equiv U_M$$

**inertial stability:**
Requires that $M$ decreases as one moves polewards on surface of constant entropy

Just as in rotating Taylor-Couette flow between concentric cylinders, in which stability requires $M$ decreasing moving towards axis of rotation

Assume the upper troposphere can be modeled as a

- homogeneous $\rho = const$,
- nearly inviscid,
- nondivergent flow $\nabla \cdot \mathbf{u} = 0$
- in a infinitesimally thin spherical shell.

## Stokes' Theorem

Circulation $C$ = line integral of velocity around closed loop = normal component of vorticity, $\omega = \nabla \times \mathbf{u}$, integrated over any surface bounded by loop:

$$C = \oint \mathbf{u} \cdot \mathbf{dl} = \int\int \omega \cdot \hat{\mathbf{n}} dA$$

If loop is latitude circle, then $C = UL$,
($L$ = length of latitude circle)

Using surface of sphere for the surface integral,
$\Rightarrow UL =$ *radial* component of $\omega$ integrated over polar cap bounded by latitude circle.

Kelvin's Circulation Theorem:

For our homogeneous, incompressible and inviscid fluid, circulation around a material loop is conserved in time.

Stokes' Theorem $\Rightarrow$ surface integral of the normal component of the vorticity is conserved.
For infinitesimal loop,

$$\frac{D}{Dt}(\omega \cdot \mathbf{n}\, dA) = 0.$$

where $\mathbf{n}$ is the normal to the loop and $dA$ is its area. For the nondivergent flow on a sphere, $dA$ is conserved $\Rightarrow$ radial component of vorticity is conserved following flow on the surface.

In solid body rotation with angular velocity $\Omega$,

$$\omega = 2\Omega$$

$\Rightarrow$ the radial component of the vorticity $= 2\Omega sin(\theta)$,
a monotonically increasing function of latitude, from south pole to north pole.

- Start in solid body rotation
- Stir the system with a *spoon* in midlatitudes such that no net zonal momentum is added to the upper troposphere.
- Consider a latitude circle outside of the stirring region. If the disturbance reaches this location, high vorticity particles move south across the latitude circle and low vorticity particles move north.
- $\Rightarrow$ equatorward flux of vorticity
- $\Rightarrow$ the radial component of vorticity integrated over the polar cap will decrease.
- Stokes' $\Rightarrow$ $U$ will also decrease
- $\Rightarrow$ The region stirred by the spoon

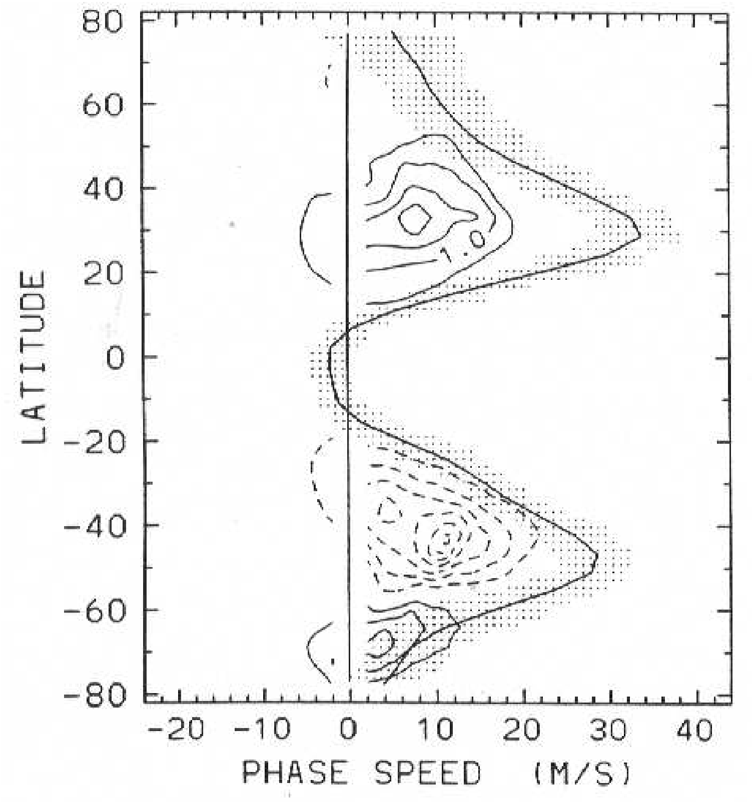
  must be accelerated
- *Angular momentum converges into the* stirred *region.*

- Only efficient mechanism for removal of momentum converging into midlatitudes is surface torques
- $\Rightarrow$ *midlatitude surface westerlies exist to remove angular momentum sucked into these latitudes in upper troposphere. Surface westerlies appear under stirred region.*
- *Ferrell cell must follow along*
- assumes one region of stirring in midlatitudes.
- as $\Omega$ is increased in Earth-like climate models, 3-cell circulation is replaced by five-cell pattern, with two regions of surface westerlies per hemisphere. "Stirring" evidently organizes itself into multiple bands when the eddy scale is much smaller than the size of the domain. On Earth, the eddy scale is large enough that we have only one band.

**Linear perspective on eddy momentum fluxes: Rossby waves**

$$\overline{u'v'} = \sum_k \int dc M(k,c)$$

$M(k,c)$ integrated over all wavenumbers at 200mb at




38°N for DJFM

- Dominated by zonal wavenumbers 4-7
- Waves transporting momentum have $c < U$.

Rossby waves exist because of the vorticity gradient. Approximating part of the sphere by a $\beta$-plane, conservation of vorticity in the rotating frame $\Rightarrow$

$$\frac{\partial \zeta}{\partial t} = -u\frac{\partial \zeta}{\partial x} - v\frac{\partial \zeta}{\partial y} - \beta v$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \nabla^2 \psi$$

$$(u, v) = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right)$$

Linearizing about the zonal flow $\overline{u} \equiv U$, we have

$$\frac{\partial \zeta'}{\partial t} = -U\frac{\partial \zeta'}{\partial x} - \gamma v'$$

where

$$\gamma = \beta - \frac{\partial^2 U}{\partial y^2}$$

$U = const \Rightarrow \gamma = \beta$, Plane wave solutions:

$$\psi = A\cos(kx + ly - \omega t)$$

$$\omega = ck = Uk - \frac{\beta k}{k^2 + l^2} \equiv \Omega(k, l)$$

$$\overline{u'v'} = -A^2\frac{kl}{2}$$

- $kl < 0 \Rightarrow$ northward momentum flux
- $kl > 0 \Rightarrow$ southward momentum flux

Meridional group velocity has same sign as $kl$

$$G_y = \frac{\partial \Omega}{\partial l} = \frac{2\beta kl}{(k^2 + l^2)^2}$$

*The meridional flux of zonal momentum in Rossby waves is in the opposite direction to the group velocity.*

- North of the source, choose $kl > 0$ so that $G_y$ is positive.
- South of the source choose $kl < 0$.

$\Rightarrow$ Rossby waves converge positive zonal momentum into the source region

Lines of constant phase ($kx + ly =$const) are tilted NW/SE for dominant equatorward propagating waves. Adding on the mean zonal flow $\Rightarrow$ "tilted trough" structure

Fluxes of (angular) momentum and vorticity are related:

For non-divergent flow

$$\zeta v = (\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y})v = \frac{\partial}{\partial x}(\frac{1}{2}(v^2 - u^2)) - \frac{\partial uv}{\partial y}$$

Averaging over $x$, and using $\overline{v} \equiv V = 0$,

$$\overline{\zeta' v'} = -\frac{\partial}{\partial y}(\overline{u'v'})$$

The zonal equation of motion in the $x$-direction is

$$\frac{\partial u}{\partial t} = fv - \frac{\partial (u^2)}{\partial x} - \frac{\partial (uv)}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Zonally averaged

$$\frac{\partial U}{\partial t} = -\frac{\partial}{\partial y}(\overline{u'v'}) = \overline{v'\zeta'} \tag{1}$$

Everything carries over to sphere, ie,

$$\overline{v'\zeta'}\cos(\theta) = -\frac{1}{a\cos(\theta)}\frac{\partial}{\partial \theta}(cos^2(\theta)\overline{u'v'}) \tag{2}$$

$\Rightarrow$ *The northward vorticity flux must sum to zero when integrated over latitude; if not identically zero, it must take on both positive and negative values.*

Return to the linearized barotropic vorticity equation, with Source $S'$ and sink $D'$

$$\frac{\partial \zeta'}{\partial t} = -U\frac{\partial \zeta'}{\partial x} - \gamma v' + S' - D'$$

$$\Rightarrow \frac{1}{2}\frac{\partial \overline{\zeta'^2}}{\partial t} = -\gamma \overline{v'\zeta'} + \frac{\overline{\zeta' S'}}{\gamma} - \frac{\overline{\zeta' D'}}{\gamma}$$

$$\frac{\partial P}{\partial t} = -\frac{\partial F}{\partial y} + \mathcal{S} - \mathcal{D}$$

$$P = \frac{1}{2\gamma}\overline{\zeta'^2} = \frac{\gamma}{2}\overline{\eta^2}$$

$$(-\gamma\eta \equiv -\zeta')$$

$$F = -\overline{u'v'}$$

*A conservation law for $P \equiv$ pseudomomentum, with $F$ = flux of pseudomomentum = Eliassen-Palm flux*

For a plane wave $\psi = A\cos(kx + ly - \omega t)$

$$P = \frac{1}{4\gamma}(k^2 + l^2)^2 A^2$$

$$\Rightarrow -\overline{u'v'} = G_y P.$$

$$\frac{\partial U}{\partial t} = -\frac{\partial P}{\partial t} + \mathcal{S} - \mathcal{D}$$

- transient, unforced:

$$\frac{\partial U}{\partial t} = -\frac{\partial P}{\partial t}$$

Conservation of (angular) momentum

$$\frac{\partial}{\partial t}\int P dy = 0$$

$\Rightarrow$ Rayleigh-Kuo criterion ($\gamma$ must change sign) for barotropic instability

- forced, dissipative, statistically steady

$$\frac{\partial U}{\partial t} = \mathcal{S} - \mathcal{D}$$

*In a statistically steady state, the eddy momentum flux convergence is determined by the sources and sinks of pseudomomentum.*

To understand form of $P$, consider fluid particles on a latitude circle, $y = y_0$, at time $t$.

Look backwards in time, to $t = 0$, and locate same particles, which trace out a curve $y - y_0 = \eta(x)$.

For simplicity suppose $\eta$ is single valued function of $x$.

The vorticity flux through this latitude, integrated from 0 to $t$, can be computed from the vorticity distribution at $t = 0$, by integrating the vorticity over the regions that end up moving through the latitude in question. If $\eta$ is small

$$\overline{\int_{y_0}^{y_0+\eta} (\zeta(0) + \gamma y) dy} = \frac{\gamma}{2}\overline{\eta^2} \qquad (3)$$

Eddy energy, $E$, unlike $P$, is not conserved by the linear evolution; eddies can exchange energy with the mean flow.

$$\frac{\partial}{\partial t}\int \frac{1}{2}\overline{u'^2 + v'^2}dy = -\int \overline{u'v'}\frac{\partial \overline{u}}{\partial y}dy$$

*Eddy energy decays as eddies propagate from regions of large zonal flow to regions of weaker flow, producing a countergradient flux of momentum.*

On the sphere, the meridional gradient of angular velocity plays the role of $\partial_y \overline{u}$:

$$\frac{\partial}{\partial t}\int \frac{1}{2}\overline{u'^2 + v'^2}dy = -\int \overline{u'v'}\cos(\theta)\frac{\partial}{\partial \theta}\left(\frac{U}{\cos(\theta)}\right)d\theta$$

Combining eddy energy and pseudomomentum equations:

$$\frac{\partial}{\partial t}\int (E - PU)dy = 0$$

$$\mathcal{E} \equiv E - PU \equiv Pseudoenergy$$

$$\mathcal{C}_0 = P \sim \eta\gamma\eta$$

$$\mathcal{C}_1 = \mathcal{E} \sim \eta(U\gamma + \gamma\nabla^{-2}\gamma)\eta$$

$$\mathcal{C}_2 \sim \eta(U^2\gamma + \gamma\nabla^{-2}U\gamma + U\gamma\nabla^{-2}\gamma + \gamma\nabla^{-2}\gamma\nabla^{-2}\gamma)\eta$$

$$\vdots$$

$$\mathcal{C}_n \sim \eta(U + \gamma\nabla^{-2})^n\gamma\eta$$

$$\mathcal{C}_2 \Rightarrow \frac{\partial}{\partial t}\int dy \left(U^2P - 2UE + \frac{\beta}{2}\overline{\psi'^2}\right) = 0$$

if $\beta = 0$, there can be no isolated "wavepacket" propagating through a region where $U$ changes appreciably, while still conserving $P$ and $\mathcal{E}$.

$\Rightarrow$ One needs more than a vorticity gradient to generate meridionally propagating Rossby waves

## *negative viscosity* – a term to be avoided

Suppose dominant wave source is in tropics, with waves propagating into midlatitudes, and being dissipated there.

then momentum fluxes into tropics,
*down* the angular velocity gradient

Term sometimes used as a synonymous for the *inverse energy cascade*. Dangerous usage, since flow of energy from eddies to mean flow is not equivalent to movement of energy to larger scales

Consider barotropic instability of a zonal jet: Energy moves from zonal flow to eddies, yet, as in any free evolution of a 2D flow, the centroid of the energy spectrum moves to larger scales as energy spreads in scale

$$\frac{\partial}{\partial t}\int k^2 \tilde{E}(k)dk < 0$$

Note: subtropical eddy momentum fluxes are upgradient with respect to the mean *angular velocity* but downgradient with respect to the mean *angular momentum*

Need theory for $D(\theta)$. If waves are linear and dissipation due to diffusion, then in limit of small diffusivity, can think in terms of *Rossby wave chromatography*

Simplest case: monotonic angular velocity profile
(ignore $\cos(\theta)$) factors for simplicity
Compute angular phase speed spectrum from of source of pseudomomentum $\mathcal{S}$,

$$\mathcal{S} \sim \int s(c)dc$$

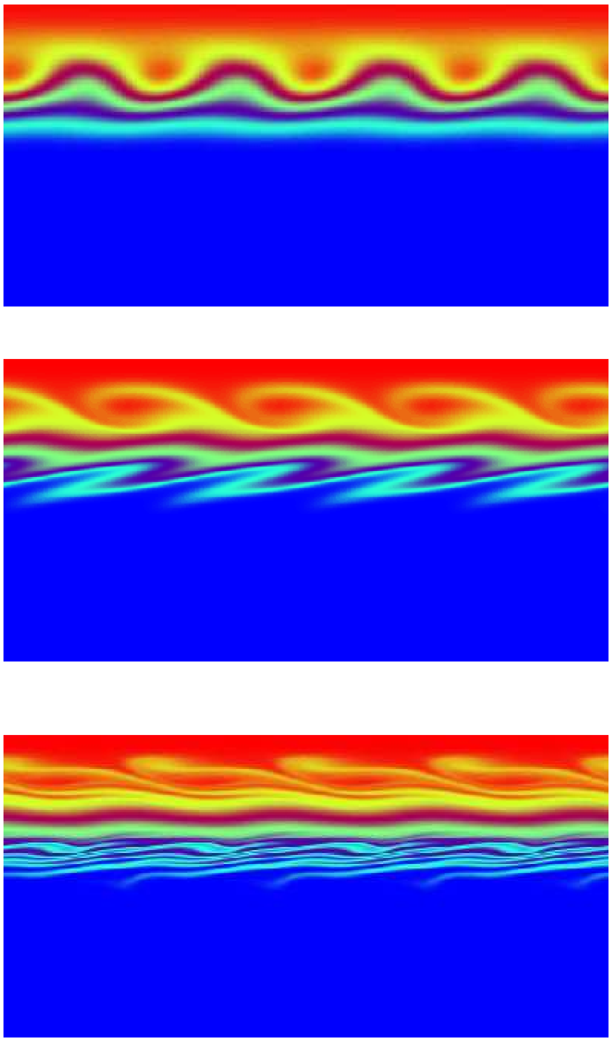
Then

$$\mathcal{D}(\theta) \sim s(U(\theta))\frac{\partial U}{\partial \theta}$$

that is, wave with phase speeds between $c$ and $c + \delta c$ dissipate and deposit their pseudomomentum in latitude band $(\theta, \theta + \delta\theta)$

In more realistic nonlinear case, waves break before reaching their critical latitude.

Consider wave with phase speed $c$. Move to reference moving with $c$. Mean flow in this frame, $U - c > 0$ for Rossby wave. But if $u' > U - c$ in part of the wave, streamlines and vorticity contours will overturn and produce mixing.



Evolution of a large amplitude (m = 4) Rossby wave, superimposed on a zonal jet similar to the mean jet in NH winter, according to barotropic vorticity equation. Positive absolute vorticity contoured.

Figure shows cyclonic roll up poleward of jet and anticyclonic roll up equatorward of jet

Two "paradigms" for wave breaking in upper troposphere (Thorncroft, et. al. (1993))

- LCI: Breaking primarily equatorward of storm track, dominated by anticyclonic roll up
- LC2: Breaking primarily poleward of storm track, dominated by cyclonic roll up

Theory for eddy momentum fluxes will require understanding factors that favor one or the other.

Orlanski (2003) shows with a non-linear shallow water model that *stronger forcing favors LC2*

Cyclones are stronger than anticyclones when forcing is strong Think of gradient wind balance, or form of vortex stretching

$$\frac{\partial \zeta}{\partial t} = -(f + \zeta)\nabla \cdot \mathbf{v}$$

Negative (anticyclonic) vorticity generated by divergence bounded by $-f$, but no comparable bound on positive (cyclonic) vorticity generated by convergence.
(Effect missing in stirred non-divergent model)

## A QG shallow water model with topography

- Rigid, flat, upper boundary
- Rigid, mountainous, lower boundary, but with mountains confined to particular band of latitudes
- Westerly jet, $U > 0$ incident on mountains

Which of the following will occur:

- $\frac{\partial U}{\partial t} > 0$. Westerlies will be accelerated by eddy momentum flux convergence in source region, as in non-divergent model
- $\frac{\partial U}{\partial t} < 0$. Westerlies be slowed down by orography

- Rigid upper boundary at $z = H_0$
- Rigid lower boundary defined by $z = h(x, y)$.
- Thickness $= H(x, y) = H_0 - h(x, y)$; thickness perturbation $H'(x, y) = -h(x, y)$.
- "Shallow water" $\Rightarrow$ small aspect ratio

  $$\frac{H_0}{L} << 1$$

  $L =$ characteristic horizontal scale of flow $\Rightarrow$ hydrostatic approximation OK

  $$\frac{\partial p}{\partial z} = -\rho_0 g$$

- Integrating from top down $\Rightarrow$ horizontal pressure gradient independent of $z$ $\Rightarrow$ we can look for solutions with horizontal velocity $\mathbf{v}$ independent of $z$.
- Vertical velocity varies linearly with $z$ from $w = \mathbf{v} \cdot \nabla h$ at $z = 0$ to 0 at $H_0$.

- Kelvin's circulation theorem remains valid (need to ignore the contribution of vertical motion to the circulation.) Stokes' for infinitesimal loop with area $\delta A$ $\Rightarrow$

$$\frac{D}{Dt}(\omega \cdot \hat{\mathbf{n}} \delta A) = \frac{D}{Dt}\left(\frac{\omega \cdot \hat{\mathbf{n}}}{H}(H\delta A)\right) = 0$$

  $H\delta A$ is mass of the vertical column of fluid whose cross section is $\delta A$;

- mass conservation $\Rightarrow$

$$\frac{D}{Dt}\left(\frac{\omega \cdot \hat{\mathbf{n}}}{H}\right) = \frac{D}{Dt}\left(\frac{f+\zeta}{H}\right) = 0$$

- Potential Vorticity

$$Q \equiv \frac{f+\zeta}{H}$$

Pressure force on the surface is normal to boundary. Slope $= \partial_x h \equiv \tan(\phi) \Rightarrow$ eastward component of pressure force $= p\sin(\phi)$. $\phi \ll 1 \Rightarrow \sin \approx \tan \Rightarrow$ zonally averaged force =

$$\overline{p_s \frac{\partial h}{\partial x}} = -\overline{h \frac{\partial p_s}{\partial x}}$$

where $p_s$ = surface pressure, related to the interior pressure $p$ by

$$p_s = p + \rho_0 g(z - h)$$

But

$$\overline{h \frac{\partial h}{\partial x}} \equiv 0,$$

$$\overline{p_s \frac{\partial h}{\partial x}} = \overline{p \frac{\partial h}{\partial x}}$$

In terms of geostrophic component of flow,

$$v_g \equiv \frac{1}{\rho_0 f_o} \frac{\partial p}{\partial x}$$

force on surface =

$$-f_0 \rho_0 \, \overline{v_g h} = f_0 \rho_0 \, \overline{v_g H}.$$

Change sign to get force exerted by surface on atmosphere:

$$-f_0 \rho_0 \overline{v_g H}$$

Assuming that mountain torque slows down westerlies, must have

$$f_0 \overline{v_g H} > 0$$

$\Rightarrow$ *poleward* mass flux

But rigid lid $\Rightarrow$ total meridional mass flux

$$\overline{vH} = \overline{v_g H} + v_a H \equiv 0$$

$\Rightarrow$ ageostrophic mass flux must be *equatorward*.

Alternative decomposition in terms of mean flow and eddies,

$$\overline{vH} = VH_0 + \overline{v'H'} \equiv 0$$

Since $v_g$ has no zonal mean, $V \equiv \overline{v_a}$.
If $v_a \ll v_g$ (i.e. if Rossby number is small) then

$$\overline{v'H'} = \overline{v_g H}$$

$\Rightarrow$

- mass flux due to eddies polewards
- mean meridional circulation equatorwards

# $\Rightarrow$ Ferrel cell !

**Quasi-geostrophic** approximation:

- $f = f_0 + \beta y$ with $\beta y << f_0$
- $\zeta << f_0$
- $h << H_0$.

Expand $Q$ in 3 small parameters

$$Q \approx \frac{f_0}{H_0} + \frac{1}{H_0}\left(\beta y + \zeta + \frac{f_0 h}{H_0}\right)$$

Define

$$q \equiv \beta y + \zeta + \frac{f_0 h}{H_0}.$$

Conservation of $Q \Rightarrow$

$$\frac{\partial q}{\partial t} = -u\frac{\partial q}{\partial x} - v\frac{\partial q}{\partial y}$$

Flow is non-divergent to first approximation, due to small Rossby number and $\beta y << f_0$. Therefore, can define a streamfunction $\psi$, as in the non-divergent case:

$$\left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}\right) = (u, v)$$

with

$$q \equiv \beta y + \nabla^2 \psi + \frac{f_0 h}{H_0}.$$

The meridional flux of QG PV:

$$\overline{v'q'} = \overline{v'\zeta'} - \frac{f_0\overline{v'H'}}{H_0} = -\frac{\partial\overline{u'v'}}{\partial y} - \frac{f_0\overline{v'H'}}{H_0}$$

Linearizing the QG PV equation about a zonally symmetric mean state,

$$\frac{\partial q'}{\partial t} = -U\frac{\partial q'}{\partial x} - v'\frac{\partial \overline{q}}{\partial y} - \mathcal{D}$$

where $\mathcal{D}$ is dissipation. The topographic source of eddies is implicit in this equation.

Multiplying by $q'$, taking the zonal average, and dividing by the mean gradient of $Q$:

$$\frac{\partial P}{\partial t} = \frac{\partial}{\partial t}\left(\frac{\overline{q'^2}}{2\gamma}\right) = -\overline{v'q'} - \overline{Dq'}/\gamma$$

where $\gamma \equiv \partial_y\overline{q}$.

In statistically steady state, in regions where $\mathcal{D} = 0, \quad \overline{v'q'} = 0.$

Suppose eddies are not dissipated over the topography. (i.e., suppose the flow is a westerly jet, then $U - c$ (with $c = 0$ in this case) is large at the latitudes of the source, and smaller to the north and south)

Assuming that the waves spread away from the source latitudes, we know that

$$\overline{v'\zeta'} = -\frac{\partial \overline{u'v'}}{\partial y} > 0$$

So not only do we have

$$-\frac{f_0 \overline{v'H'}}{H_0} < 0$$

$\Rightarrow$ poleward eddy mass flux
$\Rightarrow$ westward mountain torque on atmosphere
but these two parts of the PV flux exactly cancel!

Zonal equation of motion

$$\frac{\partial u}{\partial t} = fv - u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

or

$$\frac{\partial u}{\partial t} = (f + \zeta)v - \frac{\partial}{\partial x}\left(\frac{p}{\rho_0} + \frac{1}{2}(u^2 + v^2)\right)$$

Averaging zonally,

$$\frac{\partial U}{\partial t} = \overline{(f + \zeta)v}$$

$Ro << 1 \Rightarrow$

$$\frac{\partial U}{\partial t} = f_0 V + \overline{v'\zeta'} = f_0 V - \frac{\partial \overline{u'v'}}{\partial y}$$

Change vorticity to PV flux by adding and subtracting the mass flux:

$$\frac{\partial U}{\partial t} = f_0 V + \frac{f_0}{H_0}\overline{v'H'} + \overline{v'q'} = f_0 V^* + \overline{v'q'}$$

*Transformed Eulerian Mean (TEM) Equation of Motion*

The *residual circulation* $V^*$ is defined so as to be proportional to the total mass flux:

$$V^* = \frac{\overline{vH}}{H_0} = V + \frac{\overline{v'H'}}{H_0}$$

In our rigid lid case, $V^* = 0$, so we simply have

$$\frac{\partial U}{\partial t} = \overline{v'q'}$$

$\Rightarrow$ presence of dissipation $\mathcal{D}$ results in $\partial U/\partial t < 0$
$\Rightarrow$ mountains decelerate the westerlies everywhere!

**Two perspectives of the momentum budget**:

- Traditional:
  Momentum budget of fixed Eulerian volume bounded between
  $(y, y + \delta y)$ and $(z, z + \delta z)$

  Acceleration =

  - Coriolis force due to Eulerian mean meridional flow
  - horizontal convergence of $\overline{u'v'}$

- Transformed Eulerian Mean:
  Momentum budget of entire layer between $(y, y+\delta y)$.

  Acceleration =

  - Coriolis force due to meridional motion of center of mass
  - mountain torque
  - horizontal convergence of $\overline{u'v'}$

**A Two Layer Model**

Two isentropic layers of ideal gas separated by one interface: Pressure at the surface (constant height) varies, but pressure at the top of upper layer fixed.

Hydrostatic approximation $\Rightarrow$

$$\frac{\partial \Phi}{\partial \Pi} = -c_p \Theta; \;\; \Phi = gz; \;\; \Pi \equiv \left(\frac{p}{p*}\right)^{\kappa}$$

$\Rightarrow$

$$\begin{aligned} \Phi_2 &= c_p \Theta_2 (\Pi_B - \Pi) \\ \Phi_1 &= c_p \Theta_2 (\Pi_B - \Pi_M) + c_p \Theta_1 (\Pi_M - \Pi) \end{aligned}$$

Working in pressure coordinates, geopotential gradients are independent of pressure within each layer, which is what we need so that solutions exit in which the horizontal flow is independent of pressure, i.e., shallow water theory.

Geostrophic motion in $p$ coordinates:

$$f(u, v) = (-\frac{\partial \Phi}{\partial y}, \frac{\partial \Phi}{\partial x})$$

So zonal thermal wind is

$$f(u_1 - u_2) = -c_p(\Theta_1 - \Theta_2)\frac{\partial \Pi_M}{\partial y}$$

Radiation relaxes interface to some steep *radiative equilibrium* value.

*Kelvin's circulation theorem remains valid for an inviscid ideal gas if material loop confined to an isentropic surface.* Need only consider loops that reside entirely in one or the other layer.

$$\frac{D}{Dt}\left(\omega \cdot \hat{\mathbf{n}} dA\right) = 0.$$

Use notation

$$H \equiv \frac{\Delta p}{g}$$

for the mass per unit area of a layer.

Conservation of mass

$$\frac{D}{Dt}\left(H dA\right) = 0$$

implies

$$\frac{D}{Dt}\left(\frac{f+\zeta}{H}\right) = 0$$

Similarly, under the quasi-geostrophic approximation, in each layer,

$$q = \beta y + \zeta - \frac{f_o H}{H_o}.$$

Also the mean zonal acceleration in each layer is again given by $\overline{v(f+\zeta)}$.

QG scaling, as before, $\Rightarrow$ Transformed Eulerian Mean form

$$
\begin{aligned}
\frac{\partial U_1}{\partial t} &= fV_1^* + \overline{v_1' q_1'} \\
\frac{\partial U_2}{\partial t} &= fV_2^* + \overline{v_2' q_2'} - \kappa U_2.
\end{aligned}
$$

where we have added surface drag in the lower layer.

QG PV flux can again be written as a sum of a vorticity flux, or momentum flux convergence, and a mass flux. i.e.,

$$\overline{v_2' q_2'} = \overline{v_2' \zeta_2'} - \frac{f_o}{H_{o,2}} \overline{v_2' H_2'}$$

Using geostrophy, after a bit of manipulation one can show that

$$\overline{v_1' H_1'} = -\overline{v_2' H_2'}.$$

so eddy mass fluxes in two layers cancel.

*This is Newtons' third law in disguise:*

*Zonal force exerted by layer 1 on layer 2 is equal and opposite to force exerted by layer 2 on layer 1.*

With the interface slope $= \partial_x z_M$, the drag per unit mass exerted on the lower layer by the upper layer is

$$F = \frac{1}{H_2}\overline{p'_m \frac{\partial z_M}{\partial x}} = \frac{1}{gH_2}\overline{p'_M \frac{\partial \phi_M}{\partial x}}$$

Geostrophy $\Rightarrow$

$$f v'_2 = c_p \Theta_2 \frac{\partial \Pi'_B}{\partial x}$$

and using

$$\Phi_M = c_p \Theta_2 (\Pi_B - \Pi_M)$$

$\Rightarrow$

$$F = \frac{f_o}{gH_2}\,\overline{v'_2 H'_2}.$$

$$\sum_{i=1}^{2} \int H_i \, \overline{v'_i q'_i} \, dy = 0,$$

For linear conservative waves,

$$\frac{\partial q'}{\partial t} = -U\frac{\partial q'}{\partial x} - v'\frac{\partial \overline{q}}{\partial y}$$

$\Rightarrow$

$$\frac{\partial P}{\partial t} = -\overline{v'q'} = -\frac{\partial}{\partial y}(-\overline{u'v'}) - \frac{f_0}{H}\overline{v'H'}$$

$$P = \frac{1}{2}\frac{\overline{q'^2}}{\partial \overline{q}/\partial y}$$

$$\frac{\partial}{\partial t}\int dy(P_1 + P_2) = 0$$

$\Rightarrow$

*Charney-Stern-Pedlosky criterion for instability:*

*Mean PV gradient must take on both signs*

Nonlinear:

$$\frac{1}{2}\frac{\partial \overline{q'^2}}{\partial t} = -\overline{v'q'}\frac{\partial \overline{q}}{\partial y} - \frac{1}{2}\frac{\partial \overline{v'q'^2}}{\partial y}$$

But integrating in $y$

$$\frac{1}{2}\frac{\partial}{\partial t}\int \overline{q'^2}dy = -\int \overline{v'q'}\frac{\partial \overline{q}}{\partial y}dy.$$

- If non-conservative effects dissipate variance, then time mean PV flux must be downgradient on average.
- If advection of variance is small, PV flux is downgradient everywhere.

  $\Rightarrow$ climatic PV gradients must take both signs

$$\frac{\partial \overline{q}}{\partial y} = \beta - \frac{\partial^2 U}{\partial y^2} - \frac{f_0}{H_0}\frac{\partial H}{\partial y}.$$

If thickness gradient responsible for change in sign, we speak of *baroclinic* instability

Slope of interface sufficiently steep $\Rightarrow$ lower layer PV gradient negative.

*Statistically steady state, $\Rightarrow$ poleward PV flux in lower layer and equatorward flux in the upper layer.*

*Form drag at interface transfers zonal momentum from upper to lower layer.*

TEM equations of motion in steady state $\Rightarrow$

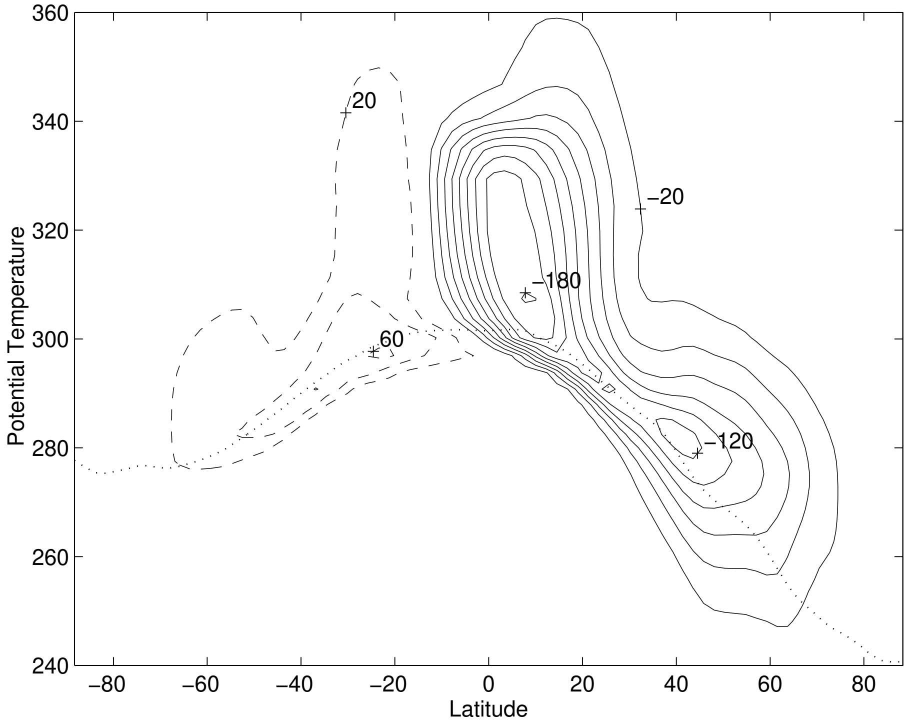
$$\begin{aligned} 0 &= fV_1^* + \overline{v_1'q_1'} \\ 0 &= fV_2^* + \overline{v_2'q_2'} - \kappa U_2. \end{aligned}$$

$\Rightarrow$ $V^*$ polewards in upper layer $\Rightarrow$ $V^*$ equatorward in lower layer, reducing interface slope

In lower layer, mass transport is in *opposite* direction to the Ekman drift associated with the surface westerlies, ie., opposite to Ferrel cell.

Eddy momentum fluxes small in lower troposphere $\Rightarrow$ form drag on the upper surface of the lower layer larger than the surface drag $\Rightarrow$ *eddy* mass flux opposite in sign, and larger in amplitude, than mass flux due to Eulerian mean.

Sharp boundary between tropics and midlatitudes has not disappeared, just because overturning is everywhere direct from the TEM perspective



Summing over the two layers,

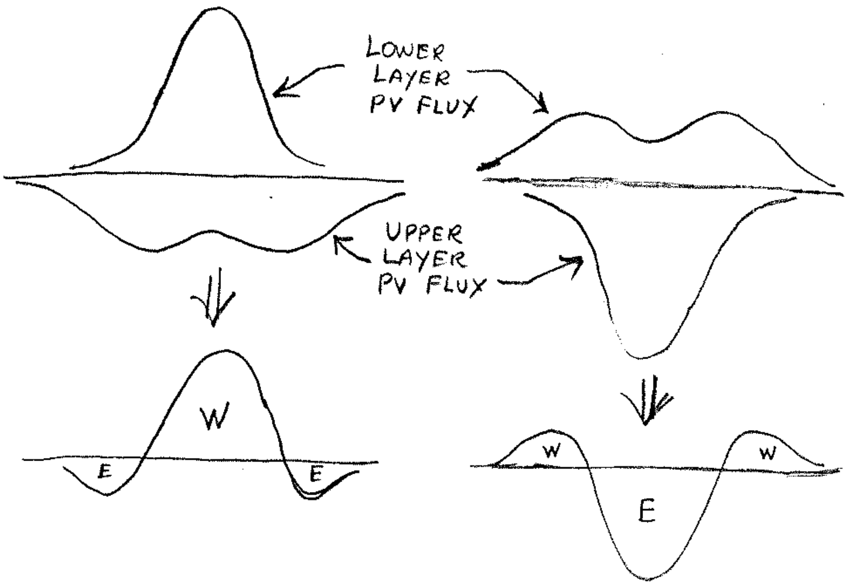
$$V_1^\star H_1 + V_2^\star H_2 = 0.$$

$\Rightarrow$

$$H_1\overline{v_1'q_1'} + H_2\overline{v_2'q_2'} = \kappa U_2 H_2$$

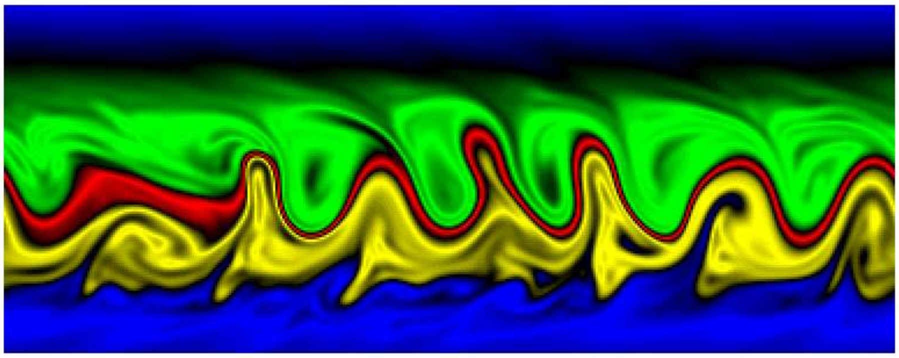
*Lower layer zonal wind determined by vertical integral of the eddy PV fluxes.*

Suppose PV fluxes are everywhere downgradient. Surface westerlies in midlatitudes requires lower layer flux to be more sharply peaked in midlatitudes, and the upper layer flux to be more broadly distributed.



Need explanation for why flux is broader in the upper layer.

- eddies more linear in the upper layer and do not dissipate as readily in source latitudes – Why?
- use $u'/(U-c)$ as breaking criterion
- baroclinic instability $\Rightarrow$
  - if $\beta = 0$ steering level in middle troposphere
  - $\beta > 0$, steering level in lower troposphere
- $\Rightarrow$ upper layer better able to support linear waves
- Positive feedback from generation of low-level westerlies: Low-level westerlies $\Rightarrow U - c$ decreases at center of the storm track in lower layer and increases in the upper layer $\Rightarrow$ increased asymmetry in breaking criterion.
- surface friction contributes in same direction



**Two-step life cycle**

- **Baroclinic growth**
  Eddies grow baroclinically, reducing vertical shear, decelerating the upper layer and creating surface westerlies through form drag.

- **Barotropic decay**
  Upper level disturbance disperses meridionally, breaks away from jet and mixes PV in relatively broad region, while lower level eddy field occludes and dissipates in situ. Upper layer deceleration partially compensated, due to dispersion of eddies, primarily into subtropics in dominant LC1 evolution, redistributing upper level deceleration equatorward.

Critical thickness gradient for baroclinic instability in two-layer model:

$$0 \sim \frac{1}{q}\frac{\partial q}{\partial y} \sim \frac{1}{f}\frac{\partial f}{\partial y} - \frac{1}{H}\frac{\partial H}{\partial y}$$

Since

$$\frac{1}{f}\frac{\partial f}{\partial y} \sim \frac{\overline{\beta}}{f} \sim \frac{\cot\theta}{a}$$

in midlatitudes the critical gradient is

$$\left.\frac{\partial H}{\partial y}\right|_c \sim \frac{H}{a}$$

$\Rightarrow$ *critical slope in this two-layer model is roughly that needed to carry one from the bottom of the model in the tropics to the top of the model near the poles*

($\Rightarrow$ cannot use two-level model to study whole planet.)

## Baroclinic adjustment

Radiative forcing is weak and slightest amount of instability is removed by eddies. $\Rightarrow$ interface slope equal to critical value.

Adding more layers, critical slope $\propto$ mean depth of lowest layer $\Rightarrow$ critical slope vanishes in continuous limit – for inviscid problem.

## Continuous stratification

- PV gradient generally positive throughout the troposphere, as in upper layer of 2-layer model.
- Charney model $\Rightarrow$ baroclinic instability arises in QG theory because of temperature gradient along surface. $\Rightarrow$ *surface* plays the role of lower layer of two-layer QG model!
- Argument applied to explain poleward mass flux in upper layer of 2-layer model seems to apply to every infinitesimal isentropic layer in the troposphere.
- Where is the return flow?
- QG theory $\Rightarrow$ in $\delta$-function at surface!

Pick a latitude, and find those isentropic layers that hardly ever reach the surface. Lower isentropes comprise the *surface layer*.
Observations suggest that essentially all of the return flow occurs in surface layer.

Depth of surface layer:

$$h_I \approx \frac{\Theta'}{\partial \overline{\Theta} / \partial p}.$$

$\Theta' \approx 10° K$, and $\partial \overline{\Theta} / \partial z \approx 3° K/km \Rightarrow$ surface layer extends through one-third of the troposphere. QG scaling assumes thickness is infinitesimal!

If

- surface layer shallow compared to scales on which the geostrophic flow varies
- static stability is uniform,

$\Rightarrow$ mass flux $\propto$ surface heat flux divided by static stability

$$\frac{1}{g} \frac{\overline{v'(\Theta_I - \Theta'_s)}}{\partial \overline{\Theta} / \partial p} = -\frac{1}{g} \frac{\overline{v'\Theta'_s}}{\partial \overline{\Theta} / \partial p}$$

$\Theta_I = const =$ top of the interrupted layer.

## The Hadley cell

Angular momentum

$$M = (\Omega a \cos\theta + u) a \cos\theta$$

pressure coordinates ($\omega \equiv Dp/Dt$):

$$\frac{\partial \overline{M}}{\partial t} = -\frac{1}{a\cos\theta}\frac{\partial\left(\cos\theta\overline{vM}\right)}{\partial\theta} - \frac{\partial\left(\overline{\omega M}\right)}{\partial p}$$

or

$$a\cos\theta\frac{\partial U}{\partial t} = -\frac{1}{a\cos\theta}\frac{\partial\left(\cos\theta V\overline{M}\right)}{\partial\theta} - \frac{\partial\left(\overline{\omega}\overline{M}\right)}{\partial p} \\ -\frac{1}{a\cos\theta}\frac{\partial\left(\cos\theta\overline{v'M'}\right)}{\partial\theta} - \frac{\partial\left(\overline{\omega' M'}\right)}{\partial p}$$

Using

$$f + \overline{\zeta} = -\frac{1}{a^2 cos(\theta)}\frac{\partial\overline{M}}{\partial\theta}$$

can rewrite

$$\frac{\partial U}{\partial t} = \left(f + \overline{\zeta}\right)V - \overline{\omega}\frac{\partial\overline{u}}{\partial p} - \frac{1}{a\cos^2\theta}\frac{\partial\left(\cos^2\theta\overline{u'v'}\right)}{\partial\theta} - \frac{\partial\overline{\omega' u'}}{\partial p}$$

Vertical eddy flux mostly important only in planetary boundary layer.

Since eddy heat fluxes are small in tropics, no need to consider TEM formalism.

Ignoring transport of relative angular momentum by the mean meridional circulation $\Rightarrow$

$$\frac{\partial U}{\partial t} = 0 = fV - \frac{1}{a\cos^2\theta}\frac{\partial\left(\cos^2\theta\overline{u'v'}\right)}{\partial\theta}$$

Low Rossby number assumption ($\zeta << f$) breaks down in low latitudes.

$Ro \ll 1 \Rightarrow V$ is slave to eddy momentum fluxes in steady state.

But in low latitudes, one expects intuitively that thermal forcing must exert a direct effect on $V$.

To simplify, drop vertical advection of momentum by mean circulation (we will return to this assumption when discussing superrotation):

$$0 = \frac{\partial U}{\partial t} = (f + \overline{\zeta})V - S$$

Overturning no longer tied to the eddy stress. Thermal forcing increases $\Rightarrow V$ increases while $f + \zeta$ decreases. Eddies do not have as much time to extract momentum from poleward moving air.

If $S = 0$, must either have $V = 0$ or $f + \zeta = 0$.
If $V \neq 0$, then angular momentum conserved along flow
$\Rightarrow$

$$U = U_m = \frac{\Omega a \sin^2(\theta)}{cos(\theta)}$$

**Reduced gravity shallow water model of upper tropospheric layer**

- Flow driven by radiative forcing = mass fluxes that relax the interface towards some radiative equilibrium.
- The pressure at the upper boundary is assumed to be constant.
- No pressure gradient in lower layer.

$$0 = \frac{\partial U}{\partial t} = (f + \zeta)V - S$$

$$0 = \frac{\partial V}{\partial t} = -fU - \frac{g^\star}{a}\frac{\partial H}{\partial \theta} - \frac{V}{a}\frac{\partial V}{\partial \theta} - U^2\frac{\tan(\theta)}{a}$$

$$0 = \frac{\partial H}{\partial t} = -\frac{1}{a\cos(\theta)}\frac{\partial}{\partial \theta}(\cos(\theta)VH) - \frac{H - H_{eq}(\theta)}{\tau}$$

Have ignored the vertical momentum transfer associated with mass transfer between layers, equivalent to neglecting the vertical advection of momentum.

For Earth-like parameter range $V \ll U$ and $U \ll fa$, $\Rightarrow$ geostrophic balance

$$0 = \frac{\partial V}{\partial t} = -fU - \frac{g^\star}{a}\frac{\partial H}{\partial \theta}$$

Need to specify $S$.
Assume $S = \kappa U$ just to be explicit,

Assume forcing ($H_{eq}$) symmetric about equator, $\Rightarrow V(0) = 0 \Rightarrow U(0) = 0$.

(If $S(0) \neq 0$ no steady state possible, without including vertical advection)

Suppose there exist point off equator with local maximum of $M$. Then $f + \zeta = 0$. $\Rightarrow$ nothing to balance $S$. $\Rightarrow U \leq U_m(\theta)$ everywhere.

When $\kappa = 0$, one solution of these equation is simply radiative equilibrium: $H = H_{eq}; U = U_{eq}, V = 0$, where

$$fU_{eq} = -\frac{g^*}{a}\frac{\partial H_{eq}}{\partial \theta}$$

If

$$H_{eq} \approx H_0(1 - \alpha\theta^2)$$

Then,

$$U_{eq} \approx \frac{\alpha\, g^* H_0}{\Omega a}$$

or

$$\frac{U_{eq}}{\Omega a} \approx \alpha R \Delta_V$$

where

$$R \equiv \frac{gH_0}{\Omega^2 a^2} \approx 0.7$$

and $\Delta_V$ is fractional change in $\Theta$ between upper and lower troposphere

Note:

Presence of static stability $\Delta_V$ in definition of $U_{eq}$ a bit misleading.

In continuous problem, suppose "radiative-convective" equilibrium $\Theta$ is

$$\frac{\Theta_{eq}}{\Theta_0} = 1 - \Delta_H \sin^2(\theta) + \Delta_V \frac{z}{H} \approx 1 - \Delta_H \theta^2 + \Delta_V \frac{z}{H}$$

Think of bottom of upper layer in reduced gravity shallow water model as surface of constant $\Theta$. Then thickness of layer in radiative-convective equilibrium

$$\frac{H_{eq} - H_0}{H_0} = -\frac{\Delta_H}{\Delta_V}\theta^2$$

$\Rightarrow \alpha = \Delta_H / \Delta_V$

$$\frac{U_{eq}}{\Omega a} = R\Delta_H$$

If $\Delta_H \approx 0.3$
(pole-to-equator temperature difference in radiative equilibrium of $100K$)

$\Rightarrow R\Delta_H \approx 0.2$

Whatever $\kappa$, must have $V \neq 0$ at least where $U_{eq} > U_M$.

$$U_M \approx \Omega a \theta^2.$$

$\Rightarrow$ circulation must exist at least from the equator to

$$\theta_H \approx \sqrt{R\Delta_H} \approx 0.5$$

In the limit $\kappa \to 0$, since $V \neq 0$ must have zero absolute vorticity, $U = U_M$, in this region.

More precisely, assume as $\kappa \to 0$ that there is a latitude $\theta_H$ equatorward of which $V \neq 0$ and $U = U_M$, and poleward of which $V = 0$ and $U = U_{eq}$.

Refer to the region with $\theta < \theta_H$ as the "tropics".

Within tropics can compute thickness gradient consistent with angular momentum conservation:

$$\frac{g^\star}{a}\frac{\partial H}{\partial \theta} \approx -2\Omega^2 a \theta^3$$

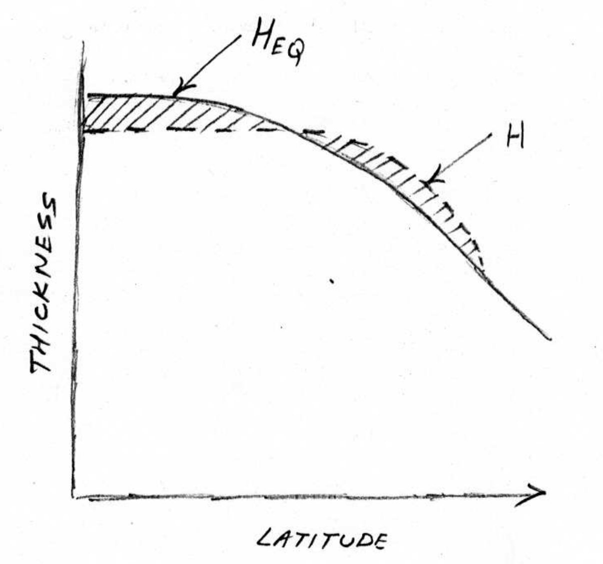
$$\frac{H(\theta) - H(0)}{H_0} = -\frac{1}{2R\Delta_H}\theta^4$$

Note flatness near equator, which keeps $U$ from increasing faster than allowed by angular momentum conservation.

Need to solve for $\theta_H$ and $H(0)$ simultaneously. The constraints are continuity of $H$ at $\theta = \theta_H$ and mass conservation:

$$\int_0^{\theta_H} (H(\theta) - H_{eq})d\theta = 0.$$

The figure below allows one to picture how these two constraints are satisfied simultaneously. Height gradient




at the boundary of the tropics will be discontinuous $\rightarrow$ U overshoots radiative equilibrium value.

In the small angle approximation, result is

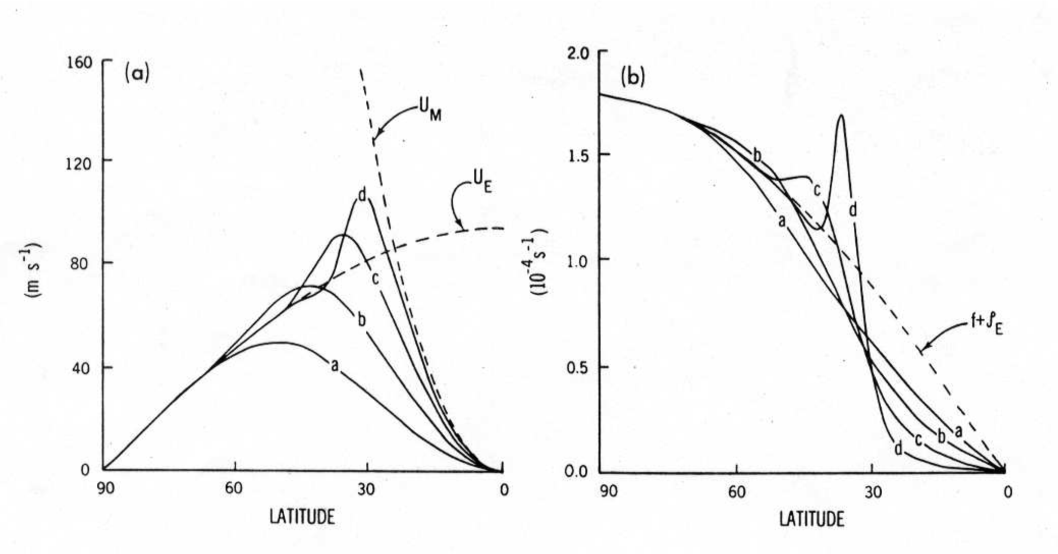
$$\theta_H = \sqrt{\frac{5}{3} R \Delta_H} \approx 0.6$$

. For $\Delta_H R \approx 0.2$ one gets $\theta_H \approx 0.6 \approx 30°$.

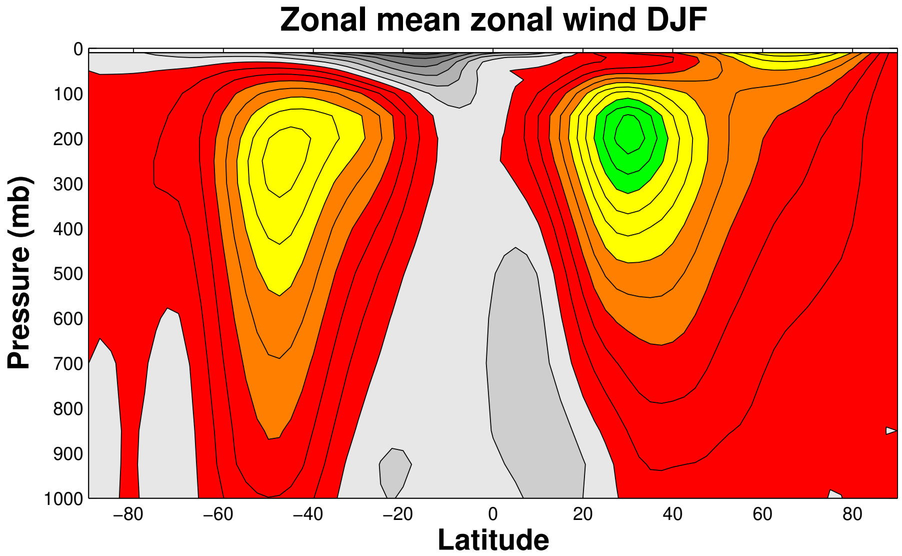
Width of circulation independent of $\tau$, the radiative relaxation time.

Mass transport $V$ proportional to $1/\tau$.

Numerical solutions with this axisymmetrtic shallow water model on the sphere, in which $H_{eq} = H_{eq}(0) - \alpha \sin^2(\theta)$ and with $S = -\kappa U$:



Eddy stresses are clearly substantial once the flow approaches the subtropics.




Open question: How should one model $S$?

Interesting limit: weak Rossby waves
No stress in tropics ⇒
zero vorticity and zero vorticity gradient in tropical upper troposphere ⇒
Rossby waves cannot penetrate into tropics
⇒ *Do Rossby waves hold themselves up by their bootstraps by creating vorticity gradient that they propagate on?*

Variations on the "inviscid" limit shallow water theory:

- heating asymmetric about equator (Lindzen-Hou)

$$\frac{H_{eq} - H_0}{H_0} = -\frac{\Delta_H}{\Delta_V}\theta^2 + \epsilon\theta$$

  $V = 0$ at $\theta_I$, the ITCZ. Need to satisfy "equal area construction" on both sides of ITCZ. Given $\theta_I$, this construction creates discontinuity in $H$ at $\theta_I$. Need to vary $\theta_I$ until $H$ is continuous. $\theta_I$ moves into warmer hemisphere.

- $H_{eq} = Q\delta(\theta)$ "ITCZ heating" at equator
  If $Q$ is chosen such that integrated mass source is the same as above, then inviscid limit unchanged

- strength of $\delta$-function heating along equator a function of $x$ (unsolved)

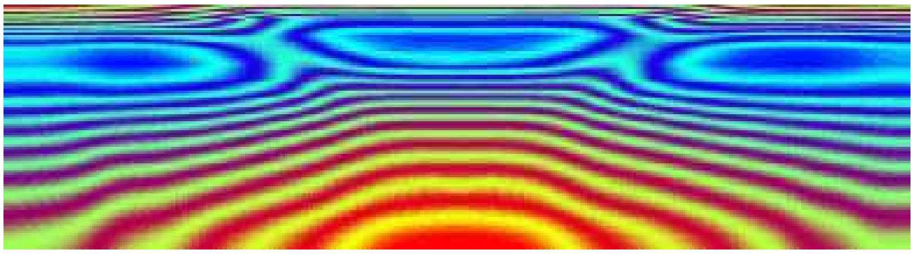
- $f$-plane, isolated $\delta$-function heating $\Rightarrow$ circular anticyclone with $f + \zeta = 0$ No flow outside of critical radius which is $\propto$ strength of heating

- $f$-plane with 2 $\delta$-function mass sources. If the two anticyclones do not touch, solution as before. If they do touch what is the solution? (unsolved)

In continuous geometry argument goes through essentially unaltered.

But does extent of region within which $U = U_M$ depend on $z$?. In a simple special case, one can argue that extent is roughly $\propto \sqrt{z}$

Comprehensive GCM, but with zonally symmetric boundary condition; Temperature:



**Traditional view of width of Hadley cell: Conserve angular momentum until flow becomes baroclinically unstable**

Use two-layer model's criterion for instability:

$$\beta - \frac{f}{H_0}\frac{\partial H}{\partial y} = 0 \Rightarrow \frac{\partial H}{\partial y} = \frac{\beta H_0}{f}.$$

From thermal wind balance,

$$g^* \frac{\partial H}{\partial y} = f(U_1 - U_2)$$

With $U_2 \approx 0$, then, in the upper troposphere

$$U_1 \approx \beta\lambda^2 = \beta\frac{g^* H_0}{f^2}$$

or

$$\frac{U}{\Omega a} \approx R\Delta_V \frac{\cos(\theta)}{\sin^2(\theta)}.$$

But

$$\frac{U_M}{\Omega a} \approx \frac{\sin^2(\theta)}{\cos(\theta)}.$$

$\theta \ll 1 \Rightarrow$ intersection at

$$\theta_{BC} \approx (R\Delta_V)^{\frac{1}{4}}$$

$$\theta_{BC}^4 = \frac{\Delta_V}{\Delta_H}\theta_H^2$$

Reconciliation of two approaches:

Width of tropics = $\mathrm{Min}(\theta_{BC}, \theta_H)$.

- $\theta_{BC} < \theta_H$:
  Alternative perspective: Let eddies transport heat while performing *Hadley adjustment* on modified temperatures. That is, eddies cool subtropics, modifying $H_{eq}$, creating larger tropical-to-subtropical gradient, resulting in stronger circulation.

- $\theta_{BC} > \theta_H$:
  Is this physically realizable? If atmosphere is heated from below, what determines static stability in between? Would static stability decrease until baroclinic instability spread to $\theta_H$?

One can use other instability criteria:

Eady or Charney growth rate is
$fRi^{-1/2} \approx fU/NH$, or

$$\omega_I \approx \frac{fU}{\sqrt{gH\Delta_V}} \approx \frac{U}{a\sqrt{R\Delta_V}}\theta$$

If we suppose that this growth rate be above some critical value $\kappa$, set by frictional damping rate perhaps, then $U_M$ reaches this value at

$$\theta_\kappa^3 \approx \frac{\kappa}{2\Omega}\sqrt{R\Delta_V}.$$

qualitative behavior is the same.

**Moist Hadley Cell**

How can we talk about Hadley cell without mentioning moist convection??

- Consider energy balance at the top of the atmosphere.
- Ignore energy transport by the ocean (and in stratosphere) $\Rightarrow$ no fluxes through the surface in steady state; Energy fluxes at the tropopause are only sources of energy for the troposphere.
- Incoming solar radiation $S(\theta)$, function of latitude only
- Outgoing infrared radiation $I(T)$, function of $T$ only.
- Ignore heat transport out of Hadley cell into mid-latitudes (or think of effect of eddy heat fluxes as specified and modifying $S(\theta)$).

$$\int_0^{\theta_H} (S(\theta) - I(T)) \cos(\theta) d\theta = 0$$

- Thermal wind balance,

$$f\frac{\partial U}{\partial z} = -\frac{g}{T}\frac{\partial T}{\partial y}.$$

- $U(0) \approx 0 \Rightarrow$

$$u(H) = -\frac{gH}{fT_0}\frac{\partial[T]}{\partial y}$$

where $[T]$ is the vertically averaged temperature. (assuming that the height of the tropopause is independent of latitude)

- Assume $I$ is function of this vertically averaged temperature, $[T]$, i.e., (assume that variations in temperature are coherent in the vertical).

- Angular momentum conservation $\Rightarrow$ $[T](\theta) - [T](0) \Rightarrow$ profile of outgoing radiation.

- Continuity of $[T]$ at the boundary of the cell and conservation of energy result in the same equal area construction for $[T](0)$ and width of the cell.

- Solution provides *total* energy transport by the circulation. It does not tell us the mass transport.

Energy conservation in a compressible fluid. Start with the conservation of kinetic and potential energy,

$$\rho\frac{D}{Dt}\frac{1}{2}|\mathbf{v}|^2 = -\rho\mathbf{v}\cdot\mathbf{v}\Phi - \mathbf{v}\cdot\nabla p$$

or

$$\rho\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{v}|^2 + \Phi\right) = -\nabla\cdot(p\mathbf{v}) + p\nabla\cdot\mathbf{v}$$

For internal energy,

$$\rho\frac{De}{Dt} = Q - p\nabla\cdot\mathbf{v}$$

Here $\Phi = gz$ is the geopotential and $Q$ is the heating rate. Total energy conservation:

$$\rho\frac{D}{Dt}\left(\frac{1}{2}|\mathbf{v}|^2 + \Phi + e\right) = -\nabla\cdot(p\mathbf{v}) + Q.$$

Flux form:

$$\frac{\partial}{\partial t}\rho\left(\frac{1}{2}|\mathbf{v}|^2 + \phi + e\right) = -\nabla\cdot\left(\rho\mathbf{v}\left(\frac{1}{2}|\mathbf{v}|^2 + \phi + h\right)\right) + Q,$$

where $h = e + p/\rho$ is the enthalpy. For our ideal gas, $e = c_v T$ and $h = c_p T$. We also define the dry static energy $s$

$$s = \phi + h = c_p T + gz.$$

Can ignore kinetic energy transport: In a steady state

$$\nabla\cdot(\rho\mathbf{v}s) = Q$$

If the dry stability of the atmosphere is positive, then the dry static energy increases with height:

$$\frac{\partial s}{\partial z} = c_p \frac{\partial T}{\partial z} + g = c_p \frac{T}{\Theta} \frac{\partial \Theta}{\partial z}.$$

Vertically integrating,

$$\nabla \cdot (V \Delta_D) = \int Q$$

where $V$ is the mass flux in either the poleward or equatorward branch of the cell and

$$\Delta_D \equiv \frac{dry\, static\, energy\, transport}{mass\, transport}$$

or the difference in dry static energy between poleward and equatorward flows. In the tropics, $\Delta_D \approx (45^\circ K) c_p$.

Vertical integral of $Q$ is sum of

- difference between radiative fluxes at top and bottom
- sensible heat flux at surface;
- latent heating.

$$\nabla \cdot (V \Delta_D) = R_T + R_B + S + LP$$

where $L$ = latent heat of condensation, $P$ = precipitation, and fluxes are positive into atmosphere.

Latent heating also related to convergence of vapor and the evaporation $E$.

$$\nabla \cdot (\rho \mathbf{v} q) = E - P$$

where $q$ is the mixing ratio

We can define *moist static energy*, $m \equiv s + Lq$, and a gross moist stability,

$$\Delta_M = m(H) - m(0),$$

so that

$$\nabla \cdot (V \Delta_M) = R_T + R_B + S + LE$$

But $R_B + S + LE = 0$ if there is no flux through the surface, so

$$\nabla \cdot (V \Delta_M) = R_T = S(\theta) - I(T)$$

In this idealized model, either

- mass transport determined by gross moist stability, given energy transport, or
- gross moist stability determined by mass transport, given energy transport

Always need some positive moist stability, even in regions of strong convection. Suppose $\Delta_M = \Delta_{MC}$ in ITCZ.

- Theory #1: Dry subtropics (Satoh, 1994)

  Subsidence rate determined by radiative cooling of atmosphere; mass transport determines $\Delta_M$ and gradient of low level moisture. (By changing humidities or surface temperature gradients?)

- Theory #2: Convective subtropics

  Simplest assumption: $\Delta_M = \Delta_{MC}$. Moist stability of convection determines mass transport

- Theory #3: intermediate

  Enough convection to prevent atmospheric subsidence from being entirely controlled by radiative cooling, but not enough for $\Delta_M = \Delta_{MC}$. Surface temperatures and humidities determined by processes other than those treated here, and

  $$\Delta_M \approx \Delta_{MC} + L(q_{ITCZ} - q) \approx \Delta_{MC} + \alpha(T^{surf}_{ITCZ} - T^{surf})$$

In reality, a lot of energy flows from atmosphere to ocean in the tropics, and this energy is removed by transport polewards in ocean.

Much of the circulation in the tropical oceans is wind-driven. Easterlies in low latitudes cause a poleward Ekman drift in the oceanic mixed layer and equatorial upwelling; circulation closed by subduction in the subtropics which returns at depth by complicated routes to supply equatorial upwelling.

- Ekman transport = mass transport in the boundary layer that results in Coriolis force that balances surface stress.
- Surface stress on the two media is by definition equal and opposite, ⇒ *Ekman mass transports are equal and opposite in atmosphere and ocean.*
- Ignore stresses on land, and assume Ekman transport is the main thing going on ⇒ mass transport in the oceanic Hadley cells comparable to that in atmospheric Hadley cell (about 60 Sverdrups in annual mean).

- Oceanic gross stability, $\Delta_O = c_O \delta T$, where $c_O$ is the heat capacity of water and $\delta T$ the characteristic temperature difference between the poleward and equatorward moving water. Adiabatic oceanic interior $\Rightarrow \delta T =$ *horizontal* temperature difference across Hadley cell.
- Ratio of the total energy transport to mass transport in either ocean or atmosphere is sum of gross stabilities, $\Delta_{tot} \equiv \Delta_O + \Delta_M$.
- In deep tropics stability primarily in ocean
$\Rightarrow$ tropical overturning more strongly constrained in coupled than in uncoupled system.

## Superrotation

"Hide's Theorem"

- Look at zonally averaged state of atmosphere
- Suppose that is local maximum of angular momentum $M$ not at ground.
- Surround this point by surface on which the mean $M$ is a constant.
- Conservation of mass $\Rightarrow$ no mean transport of $M$ into or out of volume.
- If there is any downgradient molecular (or small scale turbulent) vertical transport at all, it will remove $M$ from this region
  $\Rightarrow$ *to maintain steady state we need countergradient eddy fluxes.*
- otherwise, maximum $M$ must occur on surface, in region where near-surface $U \leq 0$.

Dominant tropospheric fluxes are down the angular momentum gradient

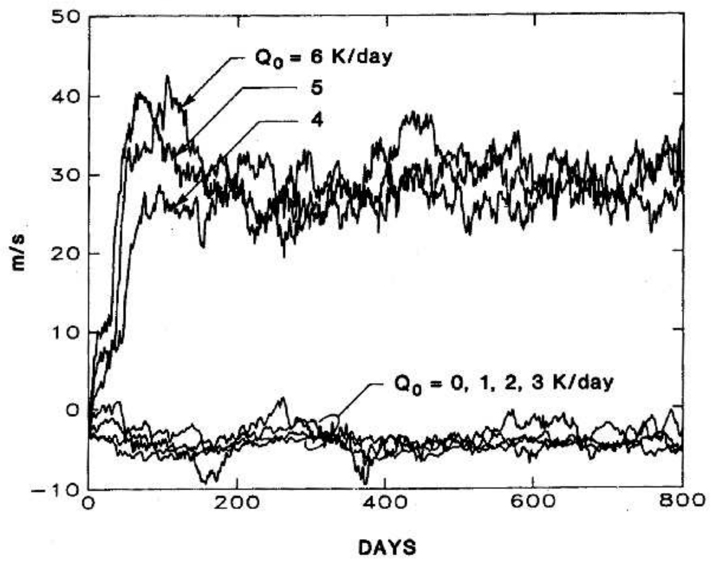
- horizontal subtropical poleward flux
- small-scale vertical flux in boundary layer

superrotation common elsewhere

- stratospheric QBO in westerly phase; eastward propagating Kelvin and/or gravity waves provide a countergradient vertical flux.
- Jupiter, Saturn, Sun

*Could the Earth's upper troposphere superrotate??*

Some idealized models do produce *abrupt* transition to a superrotating state in upper troposphere as one varies parameters. Best example from Suarez and Duffy (1992), a two-level dry model




Parameter varied controls strength of *zonally asymmetric* component of thermal forcing in tropics.

Asymmetric tropical forcing excites stationary disturbance that propagates out of the tropics in upper troposphere if conditions are favorable for Rossby waves. These waves are presumably dissipated in various complex ways once they enter midlatitudes
$\Rightarrow$ source of Rossby wave pseudomomentum in tropics
$\Rightarrow$ acceleration of the tropical flow.

Why abrupt transition?

Three feedbacks

- **feedback from stresses due to extratropical eddies**

  Midlatitude eddies that propagate deeply into tropics and break there provide drag on westerlies.

  Amount of wave breaking depends on
  $$\frac{u'}{U-c}$$
  Dominant phase speed $c \approx 5 - 12 m/s$.

  In normal state, $U(0) \approx 0 \Rightarrow$ breaking inevitable The larger U the more transparent the tropics to eastward moving extratropical disturbances, losing eddy generated stress. Abrupt change possible when $U$ rises above the typical value of $c$

- **feedback from tropically forced waves**

  For forced stationary wave to propagate out of the tropics, westerlies are required, at least in a linear picture.

  As $U$ increases, tropically forced wave will emerge more easily, providing greater acceleration near the equator, to get the ball rolling.

  Suppose one omits zonally asymmetric heating and simply imposes a torque on the zonal mean flow, varying strength of the torque.

  Is there an abrupt transition to superrotation?

  Yes, according to Saravanan (1993) using a model similar to Suarez and Duffy
  $\Rightarrow$ this feedback not essential

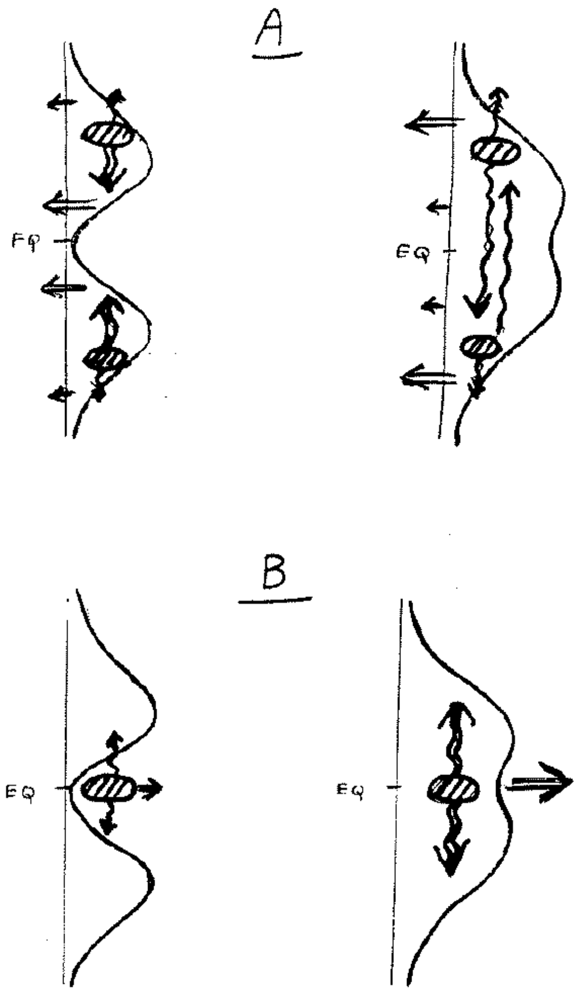



Figure 1: Schematic of the two eddy feedbacks

- **feedback from strength of Hadley cell**

  Given a positive torque at the equator, what provides the compensating deceleration in a steady state? Presumably transport of momentum from the lower to upper troposphere by the Hadley cell (or convection): With a simple dry Hadley cell,

  $$w\frac{\partial U}{\partial z}(\theta = 0) \approx torque$$

  As westerlies increase, strength of Hadley cell decreases, since upper level equatorial westerlies imply a warmer troposphere by thermal wind balance and adiabatic cooling at the equator must be reduced to be consistent with these warmer temperatures.

  It is not clear if this mechanism plays an important role in two-layer results.

  In isolation this mechanism is capable of creating a bifurcation in axisymmetric models in which there are no large scale eddy stresses, as one varies the strength of a zonally symmetric equatorial torque (Karen Shell, personal communication).

Is superrotation harder to achieve in a moist model because the Hadley cell is stronger?

Not self-evident, as there is no "large scale" upward motion in a moist Hadley cell only upward cloud-scale convective towers. Between these towers, disregarding vertical motion associated with "synoptic" eddies, one expects subsidence driven by radiative cooling.

$\Rightarrow$ no reason to believe that $w\partial U/\partial z$, where $w$ is an average over some scale (say a few hundred kilometers), is a good model of momentum transport. The net effect of the convective motions and the environmental subsidence could be a flux that is smaller than the "large-scale" advective value; environment alone transports momentum in the opposite sense!

No evidence that comprehensive climate models undergo an abrupt bifurcation of this sort.

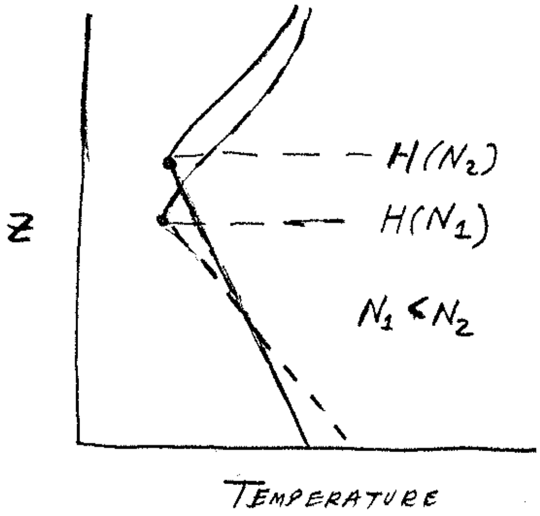
Are two-level models more likely to superrotate then models with higher vertical resolution? One can create similar behavior in multi-layer dry GCMs, but it seems to be harder to do (a good area for future work).

**The tropical tropopause**

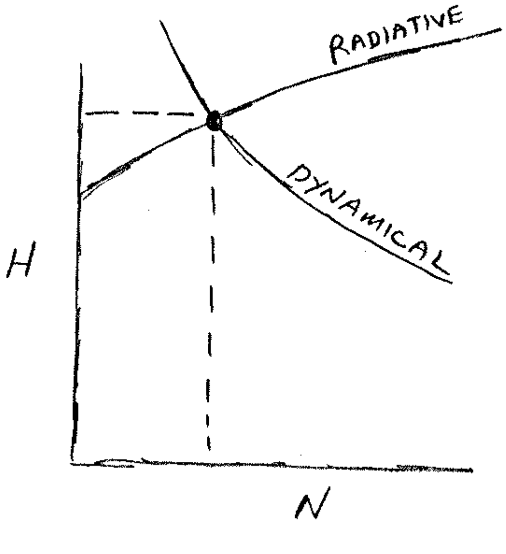
- First consider horizontally homogeneous atmosphere heated by sun.
- Pure radiative equilibrium produces a very unstable vertical temperature profile. Simple remedy: convective adjustment.
- Dry Convective adjustment: Whenever the tropospheric lapse rate exceeds the dry adiabatic value, adjust it back to this value while conserving energy; $\Rightarrow$a vertical energy flux.
- Result is "troposphere" extending from the ground up to some height $H_T$, in which the lapse rate is the dry adiabatic value, bounded above by "stratosphere" in radiative equilibrium.
- Moist convective adjustment: Adjust instead to moist adiabat corresponding to saturation at surface temperature.

Rephrase this in a form that may also be relevant in midlatitudes:

- Perform a convective adjustment, using mean lapse rate, or $N$, as a parameter. and think of resulting $H_T$ as a function of $N$. Refer to this as the "radiative constraint" between $H_T$ and $N$.



Z
H(N2)
H(N1)
N1 < N2
TEMPERATURE

RADIATIVE
DYNAMICAL
H
N

- Need a dynamical constraint so that we have two equations involving $H_T$ and $N$.
- For moist convection,

$$N^2 \approx \left(\frac{gLq(0)}{c_pT(0)}\right)\frac{1}{H_T}$$

which serves as dynamical constraint. We can think of the mixing ratio for water vapor near the surface, $q(0)$, as given, or, better, as consistent with the surface temperature predicted by radiative-convective model, assuming saturation close to the ground.

Now return to inhomogeneous tropical atmosphere with moist convection concentrated in ITCZ.

ITCZ close to moist adiabat

$\Rightarrow$ all of tropics close to same adiabat, because temperature gradients are small

Large surface gradients can still exist, with the trade wind inversion taking up the mismatch.

How close to a moist adiabat?

In a convective cloud, for a temperature difference between cloud parcel and environment of $\delta T$,
$w \propto \sqrt{R\delta T}$, which is already O(10 m/s) for 1 degree.

Can often ignore this small difference when thinking about large-scale temperature structure. But centrally important for tropical energetics.

Closely related to question of why moist convection is so sporadic, with deep cores covering $<< 1\%$ of area.

**Entropy budget argument of Emanuel, Bister, Renno, Ingersol, Pauluis, etc**

Consider horizontally homogeneous radiative-convective equilibrium.

Define system so that "external heating" is radiative cooling of atmosphere balancing the surface heat flux $Q$

Destruction of entropy by heating

$$= \frac{Q}{T_s} - \frac{Q}{T_a}$$

$T_a$ = atmospheric temperature weighted by radiative cooling. Must be balanced by creation of entropy by irreversible processes. Assume dominant process is frictional dissipation(ie. diffusion of momentum)

$$Q\left(\frac{1}{T_s} - \frac{1}{T_a}\right) = \frac{\epsilon}{T_d}$$

Temperature differences small in absolute terms $\Rightarrow$

$$\epsilon \approx Q\frac{\delta T}{T}; \quad \delta T = T_a - T_s$$

- Dry convective turbulence: fully developed turbulence $\Rightarrow$

  $$\epsilon \approx \frac{W^3}{H}$$

  $Q = 100W/m^2 \Rightarrow 10^{-2}m^2/s^3$ after dividing by mass of atmosphere.

  $\delta T/T = 0.1 \Rightarrow w^3 \approx 10$ or $w \approx 2m/s$.

- Moist convective turbulence: Assume turbulent covers fraction $\eta$ of domain, so that

$$\eta w^3 \approx 10 m^3/s^3$$

But hydrologic cycle $\Rightarrow LP = Q \approx \eta L \rho w q$, where $q$ is mixing ratio in boundary layer

$$\eta w \approx \frac{4 \times 10^{-5}}{q}$$

With $q = 0.01$, get

$$w \approx 50 m/s; \;\; \eta \approx 10^{-4}$$

If we are more careful with factors of 2, we get somewhat smaller $w$ and large $\eta$, but still somewhat extreme.

Note that $w \propto q^{-1/2}$ and $\eta \propto q^{3/2}$. Can one estimate in this way (by setting $\eta = 1$), the value of $q$ at which one makes the transition from moist to dry convective turbulence?

Is this argument OK?

Pauluis and Held (2002) argue that efficiency of moist convective "heat engine" is very poor, because dominant irreversible production of entropy is diffusion of *water*, not diffusion of *momentum*.

Estimate that efficiency reduced by factor of 10!

if this reduction $= \mu$, then replacing $\eta w^3$ by $\mu \eta w^3$, get $w \propto \mu^{-1/2}$ and $\eta \propto \mu^{1/2}$, moderating these values further.

Efficiency is so poor, perhaps should not think of atmospheric heat engine at all, but rather of an "atmospheric dehumidifier".

Can we make the same kind of entropy argument for the $extratropics$.

Yes (I think), but now

$$\frac{v^3}{L} \approx Q\frac{\delta T}{T}$$

Because turbulence is fundamentally two-dimensional, $v$ is now *horizontal* velocity scale of eddies and $L$ is *horizontal* length scale.

With $L = 10^6 m$ and $Q\,\delta T/T$ roughly as before $(10^{-3}\,m^2/s^3)$ we get $v \approx 10\,m/s$.

Without intermittency, we can now sustain eddy velocities of 10m/s everywhere because the relevant length scale is horizontal, not vertical!

Are moist midlatitude eddies efficient?

## Turbulent Diffusion in Midlatitudes

Need to be able to estimate poleward eddy heat flux in the lower troposphere to understand pole to equator temperature gradient.
closely related to mass transport in (surface layer), form drag between surface layer and free troposphere, and strength of mass transport in midlatitude isentropic overturning cell.

Try to think of this flux as simple local diffusion!

$$\overline{v'T'} \sim -\mathcal{D}\frac{\partial T}{\partial y}.$$

At the simplest level, this is just dividing the flux by the gradient, which leaves the convenient units of

$$[\mathcal{D}] \sim \frac{(length)^2}{time} = (velocity)(length)$$

. The value of diffusivity one finds in midlatitudes is

$$\mathcal{D} \approx 1 - 2 \times 10^6\, m^2/s$$

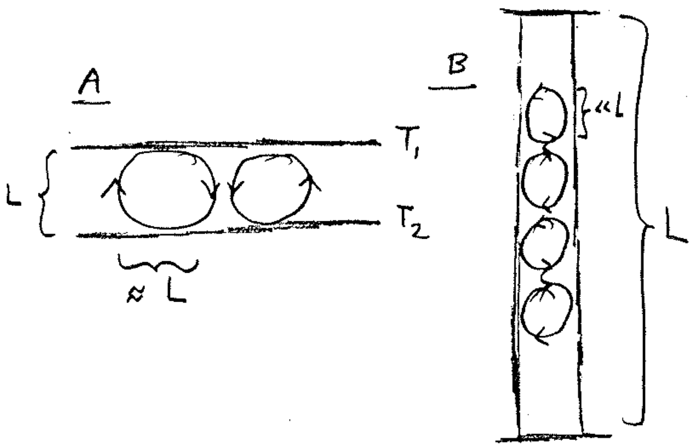
.

Is this just a convenient way of renormalizing the heat flux, or is there something fundamentally local about eddy heat transport in midlatitudes?

Benard convection: fluid confined between horizontal plates with fixed temperatures, and of horizontal extent much larger than the vertical extent.

Diffusive theories cannot have any great value for this problem because scale of eddies transporting heat is determined by vertical size of domain
$\Rightarrow$ no scale separation between eddies and mean flow
$\Rightarrow$ theory for heat transport must be fundamentally *global*.

But now consider alternative geometry in which horizontal scale of domain much smaller than the vertical scale.



In a turbulent flow, eddy scales in different directions tend to be isotropized by nonlinear advection, so the small horizontal scale should set the dominant eddy scale.

⇒ scale separation between this *mixing length* and larger vertical scale imposed by geometry.

⇒ eddy heat flux should be diffusive (although in this case, mean circulation could be dominant.)

⇒ temperature profile linear rather than homogenized in interior.

Essential distinction:

*scale of eddies is not determined by scale of the mean inhomogeneity in direction of transport, as in most familiar turbulent flows, but rather by mean flow inhomogeneity in direction perpendicular to transport.*

Why is baroclinic eddy production *diffusive* in this sense?

In linear barotropic shear flows, the zonal scale of the instability is determined by meridional scale of the mean shear variations. At finite amplitude, if sufficiently non-linear, we expect mixing across the shear will also be at this scale
$\Rightarrow$ no scale separation.

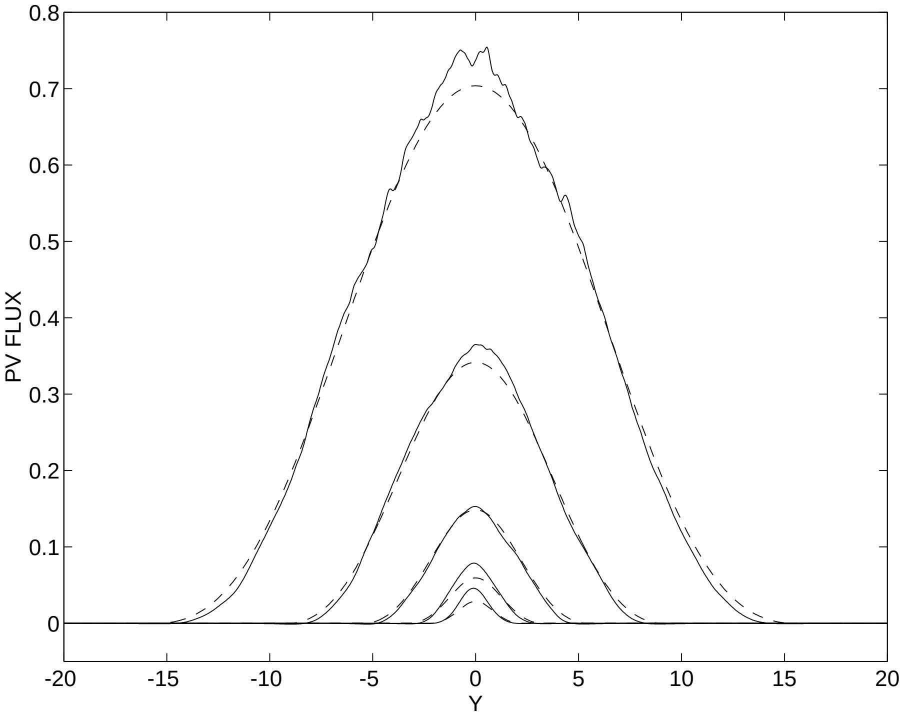
In linear baroclinic case, it is the *vertical* scale of the mean shears, multiplied by the Prandtl ratio $N/f$, that sets the zonal scale of the linearly unstable eddies.
We speak of the zonal scale as being the "radius of deformation" $NH/f$ with $H$ typically the depth of troposphere. Will nonlinear isotropization in the horizontal to turn this scale into a meridional mixing length?

If $NH/f$ is small compared to the scale of horizontal inhomogeneity, then we do have the potential for scale separation.

Even if the separation is not pronounced, we can hope, as in WKB theory, that the local theory is qualitatively useful even when the ratio of intrinsic to environmental scales is order unity.

Numerical experiments support this picture, with the important exception that the eddy scale can expand beyond the radius of deformation due to the inverse energy cascade in 2D flows.

Example of a diffusive theory being compared against a numerical two-layer QG model (Pavan and Held)



Simplest theory due to Stone: $L \sim NH/f$ and $V \sim U \sim H\partial u/\partial z$, $\Rightarrow$

$$\mathcal{D} \sim \frac{NH^2}{f}\frac{\partial u}{\partial z}$$

Why assume $V \sim U$, or eddy kinetic energy comparable to the zonal mean kinetic energy?

Baroclinic instability extracts available potential energy (APE) from mean state. Eddy of scale $L$ can only extract APE contained within domain of scale $L$. In QG theory, defining the buoyancy $b = g\Theta/\Theta_0$ we have for this energy

$$V^2 \sim APE \approx \frac{(\Delta b)^2}{N^2} \approx \left(\frac{L}{N}\frac{\partial b}{\partial y}\right)^2 \approx \left(\frac{fL}{N}\frac{\partial u}{\partial z}\right)^2.$$

$\Rightarrow$

$$V \sim \frac{L}{\lambda}U.$$

$$\mathcal{D} \sim \frac{L^2}{\lambda}U$$

If $L \sim \lambda$, (Stone) then the diffusivity is proportional to the horizontal temperature gradient, so the heat flux is proportional to the square of the temperature gradient.

Stopping the inverse cascade in 2d flow:

Following Kolmogorov/Kraichnan/Batchelor, assume inverse cascade completely determined by one parameter $\epsilon$, the rate of transfer of energy upscale: $\epsilon \sim L^2/T^3$

Suppose that we are given $\epsilon$, perhaps by thinking in terms of $Q\delta T/T$.

- Linear damping of momentum

$$\frac{\partial v}{\partial t} = -\kappa v$$

$\Rightarrow$

$$L \sim \epsilon^{1/2}\kappa^{-3/2};\ \ V \sim \epsilon^{1/2}\kappa^{-1/2}; \mathcal{D} \sim \epsilon\kappa^{-2}$$

- Nonlinear damping of momentum

$$\frac{\partial v}{\partial t} = -C|v|v$$

$\Rightarrow$

$$L \sim C^{-1};\ \ V \sim \epsilon^{1/3}C^{-1/3}; \mathcal{D} \sim \epsilon^{1/3}C^{-4/3}$$

$C = C_D/H$, where $C_D \approx 10^{-3}$ - nondimensional drag coefficient and $H =$ depth of troposphere.

$\Rightarrow$

$$L > \lambda \ if\ C_D < \frac{f}{N}!$$

- $\beta$-effect
$$\frac{\partial \zeta}{\partial t} = -\beta v$$
As scales expand, Rossby wave phase speeds increase until they are comparable to eddy velocity (Rhines scale) $\Rightarrow$
$$L \sim \epsilon^{1/5}\beta^{-3/5}; \;\; V \sim \epsilon^{2/5}\beta^{-1/5}; \mathcal{D} \sim \epsilon^{3/5}\beta^{-4/5}$$
$\Rightarrow V \sim \beta L^2$

Recent paper by Barry et al (2002) finds that $\mathcal{D} \sim \epsilon^{3/5}\beta^{-4/5}$ works pretty well in GCMs if we compute $\epsilon$ from solution.

In QG theory we can go further by setting $\epsilon$ equal to generation of APE:
$$\epsilon \sim \frac{\overline{v'b'}}{N^2}\partial\bar{b}/\partial y = \frac{\mathcal{D}}{N^2}\left(f\frac{\partial U}{\partial z}\right)^2 = \frac{\mathcal{D}}{T_e^2}$$
where
$$\frac{1}{T_e} = \frac{f}{\sqrt{Ri}}$$
is the Eady, or Charney, growth rate.

Now have 2 equations in 2 unknowns, $\mathcal{D}$ and $\epsilon$.

- $\beta$-effect

$$\epsilon \sim \frac{\mathcal{D}}{T_e^2} \; and \; \mathcal{D} \sim \epsilon^{3/5}\beta^{-4/5}$$

$\Rightarrow$

$$\mathcal{D} \sim \frac{1}{\beta^2 T_e^3} \; and \; \epsilon \sim \frac{1}{\beta^2 T_e^5}$$

- nonlinear drag

$$\mathcal{D} \sim \frac{1}{C^2 T_e} \; and \; \epsilon \sim \frac{1}{C^2 T_e^3}$$

Stopping the cascade with $\beta$ rather than drag results in a heat flux that is much more sensitive to temperature gradients.

## The Extratropical Static Stability

How high does the eddy potential vorticity mixing extend?

I.e., What determines the height of the tropopause?

Consider a Boussinesq atmosphere with no upper boundary, uniform $N(z)$ and uniform vertical shear $U = \Lambda z$ on a $\beta$-plane. PV gradient in interior $= \beta$
What is the horizontal scale of the most unstable wave?

$$h \approx \frac{f^2\Lambda}{\beta N^2}; \;\; L \approx \frac{f\Lambda}{\beta N}$$

Can obtain this result by non-dimensionalizing the linear problem, or by thinking about counter-propagating Rossby waves: finding that height $h$ at which a Rossby wave with scale $Nh/f$ would propagate westward with the right phase speed so as to match the eastward propagation of the Eady edge wave.

Vertical penetration proportional to vertical shear,
*The eddies adjust themselves so that fractional variations in* $\Theta$ *in vertical over scale of wave are comparable to variations in horizontal over scale of the planet*

Analogous (?) barotropic problem:
eastward point jet on a $\beta$-plane: $U = \Lambda|y|$.

Most unstable wave has scale $\Lambda/\beta$ in $x$ and extends roughly same distance in $y$. The baroclinic result is identical but scaled by Prandtl ratio $f/N$.

Is this scale relevant for statistically steady state?
What is the minimum distance to which vorticity must be homogenized to insure that the vorticity gradient is not negative?

By homogenizing vorticity to $y = \Lambda/\beta$ and assuming continuity of $U$ we can remove the negative vorticity gradient, but then $U$ increases everywhere. To conserve momentum, have to mix to $1.5\Lambda/\beta$. (*barotropic adjustment*).

Is the same picture relevant for baroclinic case? Can we use this as *dynamical constraint*, with $H \propto N^{-2}$, combined with the radiative constraint, to determine both $N$ and $H$?

Alternative theory (Juckes)

Assume that convection, even if it occurs in only a small portion of baroclinic eddies, is more important for maintaining static stability than the large-scale fluxes.

$\Rightarrow$ *one is always convecting in the warm sector of the surface cyclone* Then, in the dry case,

$$H\frac{\partial\Theta}{\partial z} \approx \Theta'$$

which could serve as a dynamical constraint (easily extended to the moist case).

*Confession:*

*I find the problem of how best to think about the extratropical static stability to be one of the most difficult aspects of general circulation theory*