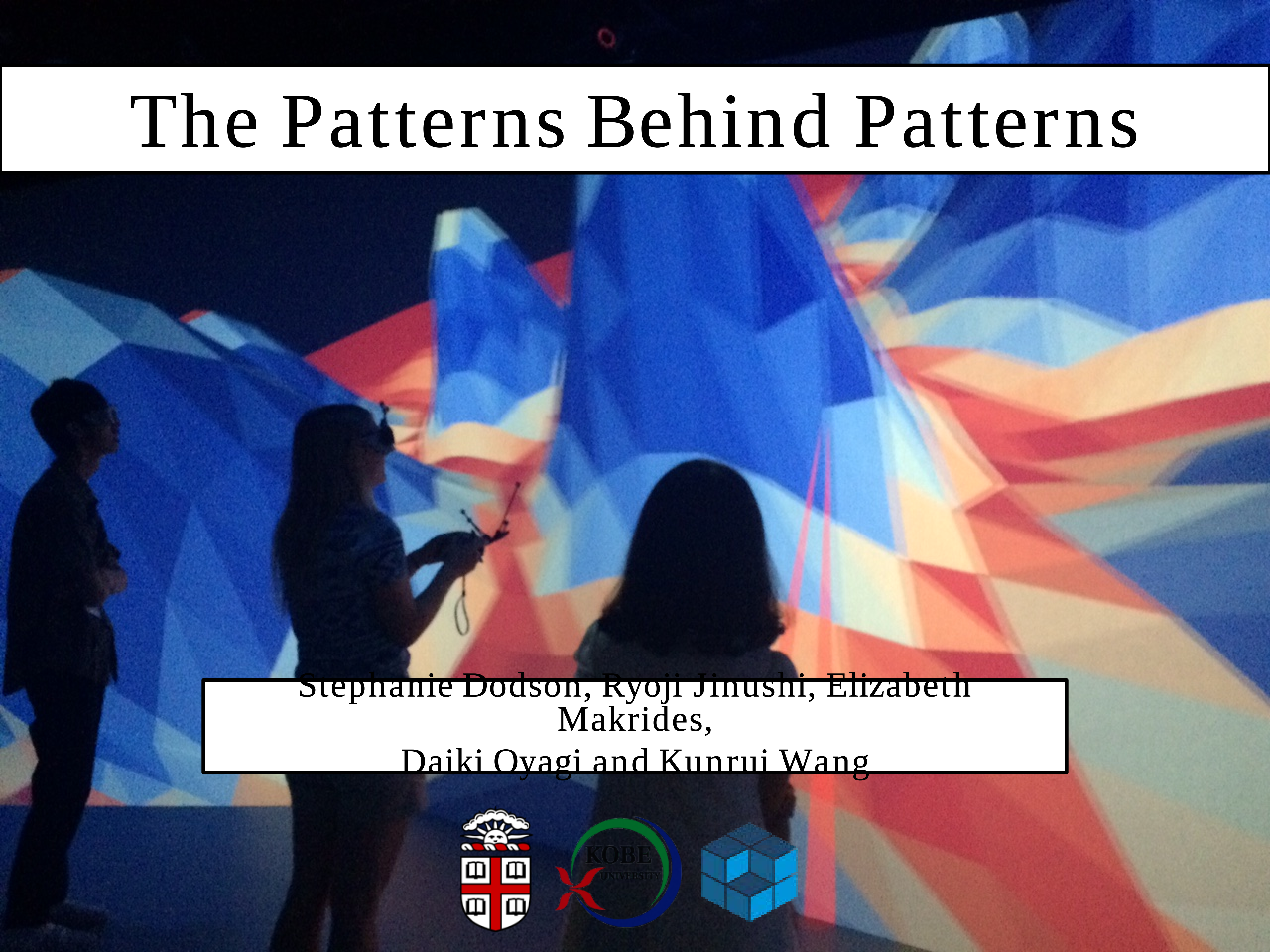
The Patterns Behind Patterns



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We say “ありがとう”

- Thank you to our hosts at Kobe University, in particular Usui-sensei, Takahashi-sensei and Miyake-sensei
- Thank you also to Kaya-sensei who initiated this summer school
- And special thanks to Meno-sensei and Morishita-sensei for their assistance with 3D visualizations here and in the CAVE

Introduction to Pattern Formation

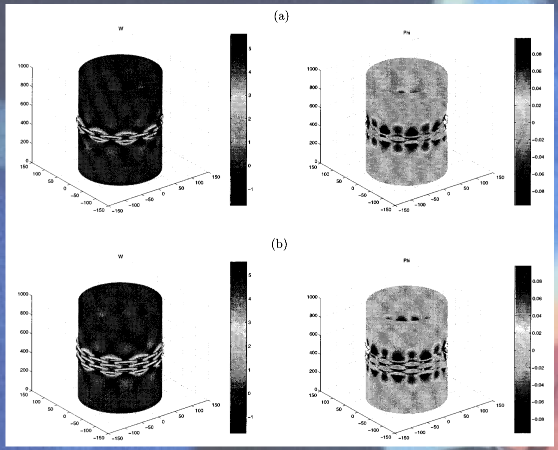
In our project we explore localized patterns on the plane

- These patterns appear in a variety of physical contexts, such as fluid systems, buckling and vegetation growth
- Understanding these patterns may lead to advances in our ability to predict and control physical processes such as crystallization

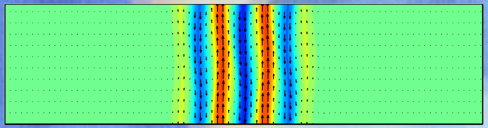
Patterns are solutions to partial differential equations (PDEs) and include spots and stripes, rhomboids and hexagons

Examples from Physical Systems

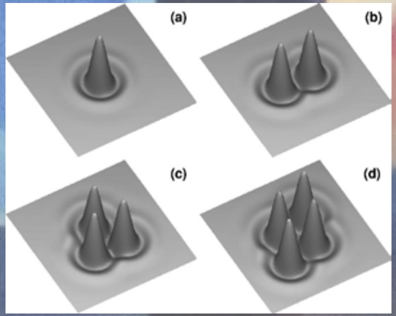
1 Bucking in thin structures[1]



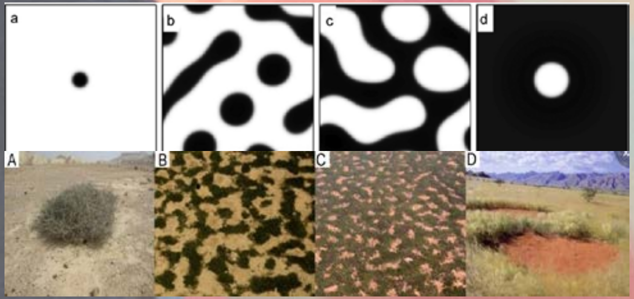
2 Plane Couette flow[2]



3 Nonlinear optics: cavity solitons[3]



4 Vegetative growth patterns[4]



1. Hunt *et al.*, 2000; 2. Schneider *et al.*, 2010; 3. McSloy *et al.*, 2002; 4. Meron, 2012.

Our Model System

The Swift-Hohenberg equation

$$U_t = -(1+\Delta)^2 U - \mu U + \nu U^3 - U^5$$

$$U = U(x,y,t), \;\; \Delta = \partial_x^2 + \partial_y^2, \;\; (\mu,\nu) \in \mathbb{R}^2$$

- For stationary solutions, $U_t = 0$
- We can vary the parameters μ and ν and also change the nonlinearity to model different physical systems
- Finally, we can consider different boundary conditions (periodic, Neumann or combination)

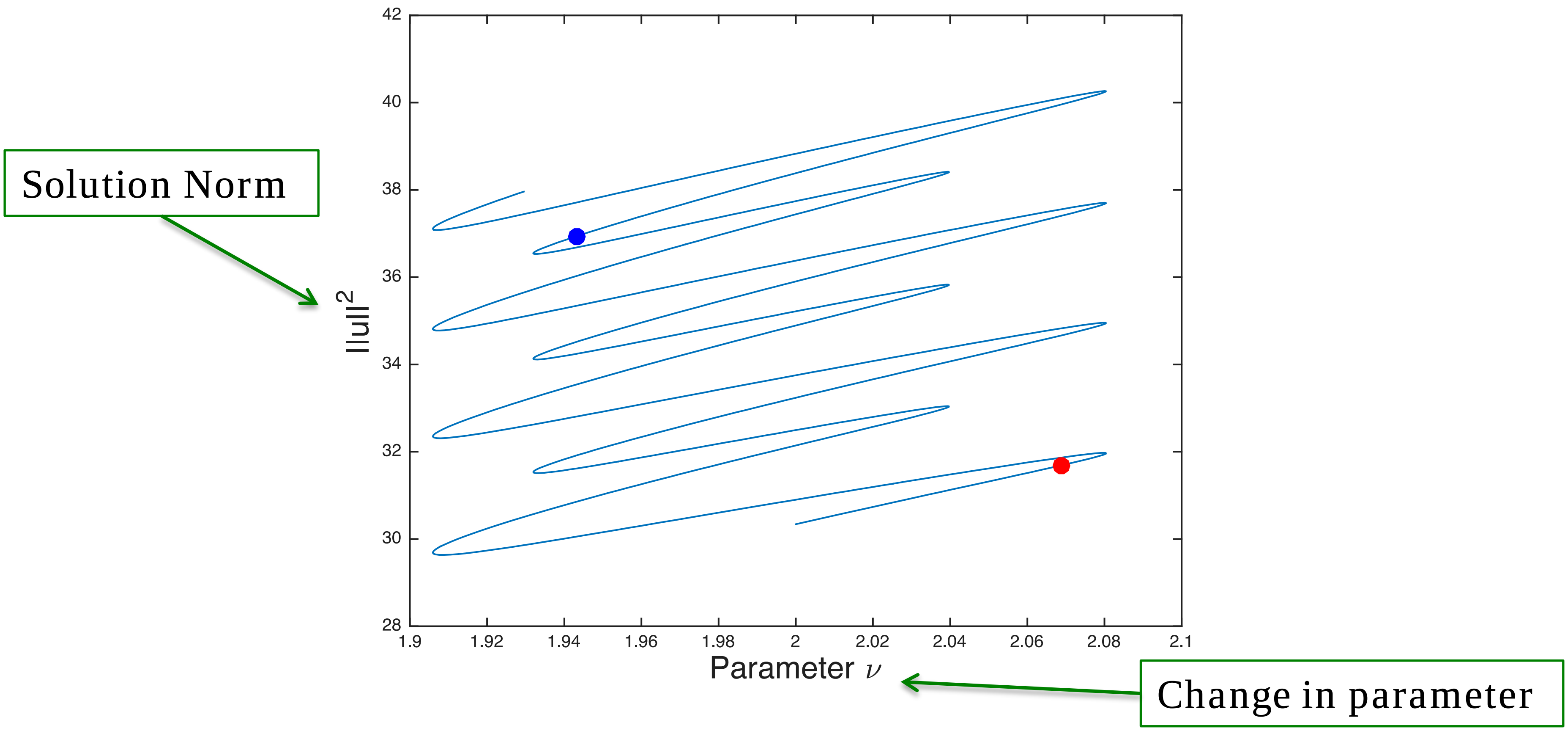
Meaning of Equation

The Swift-Hohenberg equation

$$U_t = -(1+\Delta)^2 U - \mu U + \nu U^3 - U^5$$

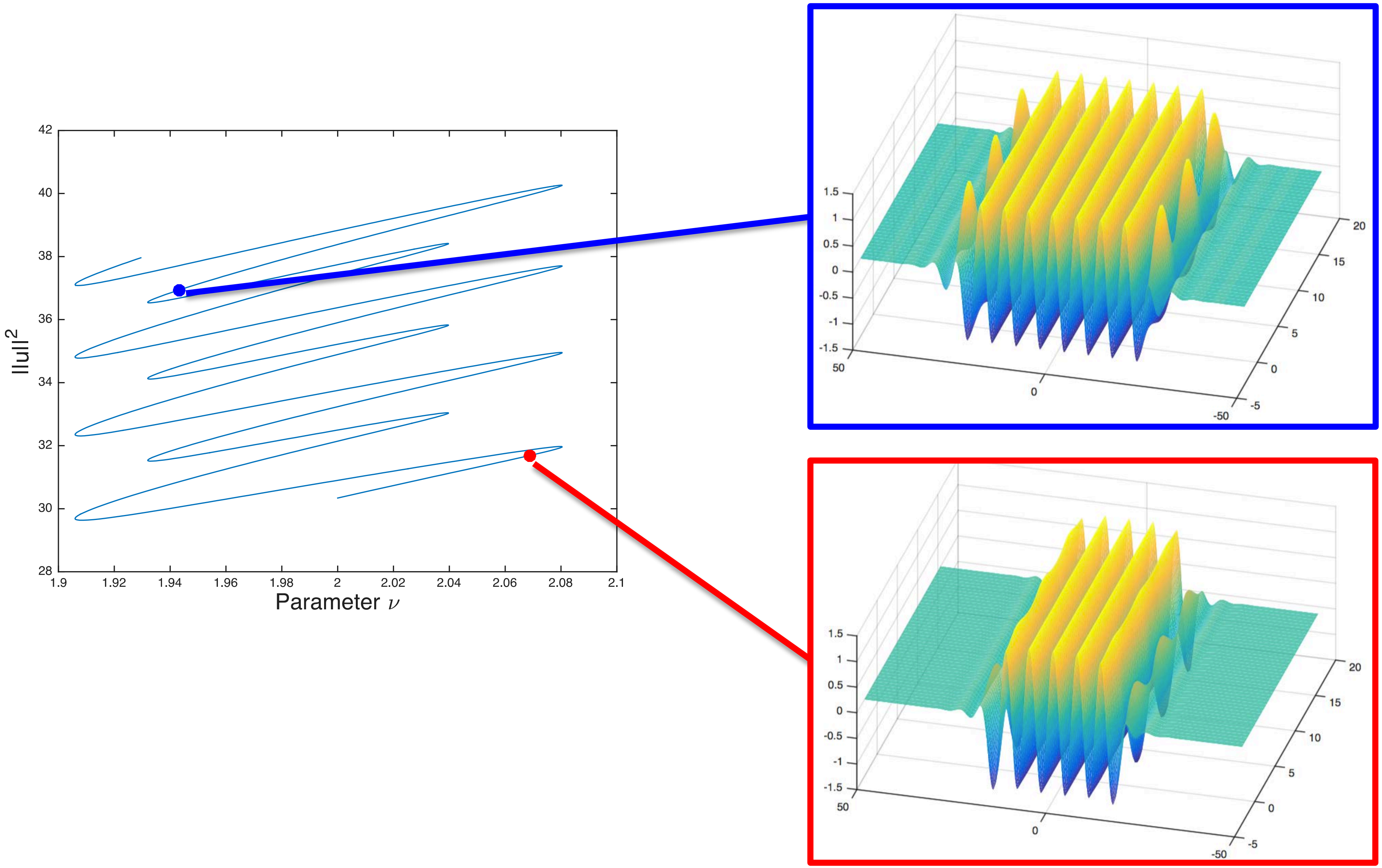
https://www.youtube.com/watch?v=_IYHT5jAInc

Main Tool: Bifurcation Diagram



- Each point corresponds to a solution
- Shows how change in parameter affects solution

Main Tool: Bifurcation Diagram



Numerical Solutions

- Computing the solutions numerically:
  - Need to discretize the operators
  - Impose boundary conditions (BC)

$$L = (Id + D^2)^2$$

$$D^2 = \begin{pmatrix} -2 & 2 & & & \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & -2 & 1 \\ & & & 2 & -2 \end{pmatrix}$$

2nd order Finite Differences (FD) with Neumann BC

$$(D)^2_{jk} = \begin{cases} \frac{1}{4}(-1)^{j+k}N + \frac{(-1)^{j+k+1}}{2\sin^2\left(\frac{(j+k)\pi}{N}\right)} & j \neq k \\ -\frac{(N-1)(N-2)}{12} & j = k \end{cases}$$

Fourier (spectral) with periodic BC on *N* grid points

Two Cases Studied Here

Nonlinearity 1:

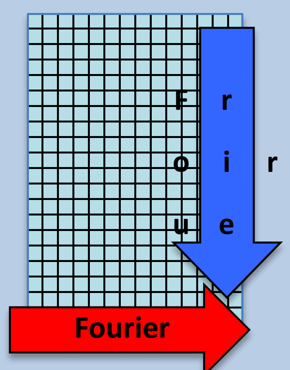
$$g_1(u) = \nu u^2 - u^3$$

Patterns:

- Rhomboids
- Hexagons

Discretization:

- Fourier – periodic
- Fourier – periodic



Nonlinearity 2:

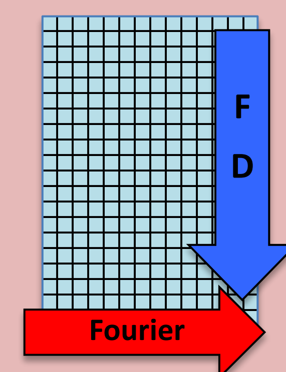
$$g_2(u) = \nu u^3 - u^5$$

Patterns:

- Stripes and spots
- All stripes or spots

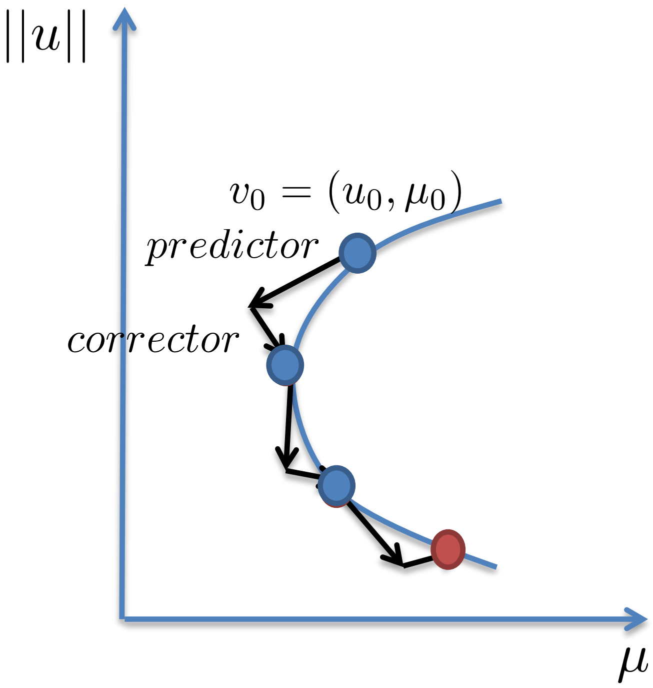
Discretization:

- Fourier – periodic
- FD – Neumann



Numerical Continuation

- Formulate the problem as
  $$f(v) = 0, v = (u, \mu), f : \mathbb{R}^{n+1} \to \mathbb{R}^n$$
- Our task: given an initial solution at $v_0 = (u_0, \mu_0)$ find a manifold of solutions connected in parameter space
- We proceed via a series of predictor-corrector steps
- Predictor: find a vector v_* in the null space of $Df(v_0)$ and make step hv_*
- Corrector: use a root finding method in the constrained space
  $$S := \{v : \langle v - (v_0 + hv_*), v_* \rangle = 0\}$$
  which is orthogonal to the predictor



Resource use across continents

Visualizations conducted using the π-CAVE and resources at Kobe University

Computations completed on Oscar, the supercomputer at Brown University

And now our results … back to the videos!