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# Lecture 3: Jets – stability and nonlinear stability

- Gulf Stream, Kuroshio, North Brazil Current, Agulhas Current, ...
- Meandering
  - Quasigeostrophic: Now, we will go back to the QG two-layer system
  - Stability
  - Appendix A. Stability Program
- Contour dynamics
  - Numerical results
  - Conclusions: numerical
  - Pulses
  - Continuum modes
  - Zero lower layer flow
- Ring formation
- Diagnostics of baroclinic amplification

### 1 —Gulf Stream, Kuroshio, North Brazil Current, Agulhas Current, ...

The wind-driven circulations result in intensified currents on the western boundaries of the ocean, related with the asymmetries produced by the beta-effect (the sphericity of the earth and the westward propagation of Rossby waves) as realized by Stommel (1948). In essence, the wind puts in negative vorticity, and, in the mid-ocean where the large-scale relative vorticity is negligible, the fluid responds by decreasing $f$: moving equatorward. The return poleward flow has to be in a boundary layer which can inject cyclonic vorticity back into the flow. This can happen by side-wall or bottom friction on the western sides of the oceans.

These major currents are several orders of magnitude larger than the mean flows (but not than the mesoscale eddies) and detach from the boundaries in various ways to form free jets. These characteristically meander and form detached eddies (like the cut-off lows in the atmosphere). We shall discuss the dynamics of meandering and eddy formation.

## 2 —Meandering

There are several models which have been applied to analyze the meandering of oceanic inertial jets:

- instability theory (almost always linear)
- thin-jet models which assume the length scale of the meanders is large compared to the width of the jet
- contour-dynamic models which follow PV contours

We will concentrate on the last one (which uses QG) , but consider first the long wave instability in the full two-layer model (Flierl, 1999). The upper and lower layer velocity perturbations have the form $u_j \exp(\imath kx - \imath kct)$, $\imath kv_j \exp(\imath kx - \imath kct)$ with the perturbation upper layer thickness and deep pressure also proportional to $\exp(\imath kx - \imath kct)$. The perturbations satisfy

$$
\begin{aligned}
(U-c)u_1 + (U_y - f)v_1 + g'h + \phi &= 0 \\
fu_1 + g'h_y + \phi_y &= k^2(U-c)v_1 \\
u_1\overline{h} + (v_1\overline{h})_y + (U-c)h &= 0 \\
-cu_2 - fv_2 + \phi &= 0 \\
fu_2 + \phi_y &= -k^2 cv_2 \\
u_2(H-\overline{h}) + [v_2(H-\overline{h})]_y + ch &= 0
\end{aligned}
$$

(with $\phi$ the lower-layer pressure). The lowest order solution in the interior region (where $c \sim U$) is found by dropping the $k^2$ terms and recognizing that long waves look like displacements of the jet plus the flow necessary to shift the axis: $u_1 = -U_y$, $v_1 = U - c$, $g'h = -g'\overline{h}_y = fU$, $u_2 = 0$, $v_2 = -c$, and $\phi = -fc$.

Operate by multiplying the $y$-momentum equation by $\overline{h}$, the mass equation by $f$, subtracting, and integrating over $y$. We do this in each layer to find

$$
\begin{aligned}
c[\overline{h}u_1]_{-\infty}^{\infty} - \int \phi\overline{h}_y + fc\int h &= k^2 \int \overline{h}(U-c)v_1 \\
&\approx k^2 \int \overline{h}U^2 \\
c[(H-\overline{h})u_2]_{-\infty}^{\infty} + \int \phi\overline{h}_y - fc\int h &= k^2 \int (H-\overline{h})(-c)v_2 \\
&\approx 0
\end{aligned}
$$

where the approximate versions of the right-hand sides are obtained by substituting the lowest order solution and assuming that $c$ is small.

These interior solutions do not decay away from the jet; therefore, we must look for decaying exterior solutions and match them to the interior forms. We have two cases to consider: the two-layer model and the equivalent barotropic model.

TWO-LAYER CASE: The interior $v$ field becomes barotropic at the wings of the jet, so we must seek a barotropic, trapped wave in the exterior. We find the required matching condition by summing the two equations

$$c[\overline{h}u_1]_{-\infty}^{\infty} + c[(H - \overline{h})u_2]_{-\infty}^{\infty} \approx k^2 \int \overline{h}U^2 \tag{2.1}$$

The barotropic external field to the north looks like:

$$\begin{aligned} u_1\overline{h} + u_2(H - \overline{h}) &= -ckH \exp(-ky) \\ v_1\overline{h} + v_2(H - \overline{h}) &= -cH \exp(-ky) \\ H(g'h + p) &= -ckH(c + f/k) \exp(-ky) \end{aligned}$$

where the amplitude is set by matching to the interior $v$ field. Similar solutions apply on the southern side. If we substitute these forms in the matching condition (2.xx), we find

$$c^2 = -\frac{k}{2H} \int \overline{h}U^2$$

Thus, $c \sim \imath k^{1/2}$ and the jet is unstable. Furthermore, the result (2.1) that the net momentum transport is related to the external barotropic wave field applies on the $\beta$-plane, since the jet width is small compared to $f/\beta$. The implication is that the external BT field will be central in determining the instability properties even on the $\beta$ plane.

EQUIVALENT BAROTROPIC CASE: When the lower layer is motionless, the exterior, trapped solutions are baroclinic with

$$\begin{aligned} u_1 &= -c\ell \exp(-\ell y) \\ v_1 &= -c \exp(-\ell y) \\ g'h &= -cf \exp(-\ell y) \end{aligned}$$

having a meridional decay rate

$$\ell^2 = f^2/g'\overline{h}$$

which is different on the northern and southern sides of the jet. The upper layer integrated equation (with $\phi$ dropped, of course)

$$c[\overline{h}u_1]_{-\infty}^{\infty} + fc \int h \approx k^2 \int \overline{h}U^2$$

simplifies because the boundary term is order $c^2$ and $c$ is now order $k^2$. Thus, the boundary terms are negligible, and

$$c = k^2 \frac{\int \overline{h}U^2}{f \int h} = k^2 \frac{\int \overline{h}U^2}{f[\overline{h}(-\infty) - \overline{h}(\infty)]}$$

as in Cushman-Roisin, et al. (1993).

The differences show up in QG theory (as we shall see) and can be traced to the $k$ structure of the Green's function: for the barotropic mode

$$G(y-y') = -\frac{1}{2k}\exp(-k|y-y'|)$$

is order $1/k$, whereas the baroclinic modes are

$$G(y-y') = -\frac{1}{2K}\exp(-K|y-y'|) \quad , \quad K = \sqrt{k^2+\gamma^2}$$

is regular and leads to $c$ order $k^2$ (the westward propagation nearly balances the eastward mean advection).

*2.1 — Quasigeostrophic:*

Now, we will go back to the QG two-layer system with both layers active. Defining the Green's function matrix by

$$\begin{pmatrix} \nabla^2 - h_2\gamma^2 & h_2\gamma^2 \\ h_1\gamma^2 & \nabla^2 - h_1\gamma^2 \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \begin{pmatrix} \delta(\mathbf{x}-\mathbf{x}') & 0 \\ 0 & \delta(\mathbf{x}-\mathbf{x}') \end{pmatrix} \tag{2.2}$$

with $h_j$ the fractional thickness of the $j^{th}$ layer and $\gamma = f_0\sqrt{(H_1+H_2)/g'H_1H_2}$ the inverse of the deformation radius. This has the property that $h_2G_{21} = h_1G_{12}$ (required for showing that $\frac{\delta H}{\delta q_i} = -\psi_i$.)

The Hamiltonian is

$$H = -\frac{1}{2}\sum_{i,j=1,2}\iint d\mathbf{x}\,d\mathbf{x}'\,h_i\,q_i(\mathbf{x})G_{ij}(\mathbf{x}-\mathbf{x}')q_j(\mathbf{x}')$$

and the bracket is

$$\{F,G\} = \sum_{i=1,2}\frac{1}{h_i}\int d\mathbf{x}\,q_i\left[\frac{\delta F}{\delta q_i},\frac{\delta G}{\delta q_i}\right]$$

*2.2 — Stability*

If we split the PV into a zonal basic state and perturbations,

$$\frac{\partial}{\partial t} q' = [\overline{q}, \psi'] + [q', \overline{\psi}] + [q', \psi']$$

with the last term dropped. If we define the displacements of PV contours by

$$q' = -\overline{q}_y \eta$$

then

$$\frac{\partial}{\partial t}\eta = -U\frac{\partial}{\partial x}\eta + \frac{\partial}{\partial x}\psi' \quad , \quad \psi' = -\int d\mathbf{x}' G(\mathbf{x}-\mathbf{x}')\overline{q}_y(y')\eta(\mathbf{x}')$$

or

$$\frac{\partial}{\partial t}\eta = -U\frac{\partial}{\partial x}\eta - \int d\mathbf{x}' G(\mathbf{x}-\mathbf{x}')\overline{q}_y(y')\frac{\partial}{\partial x'}\eta(\mathbf{x}') \tag{2.3}$$

We can write this as a symmetric kernel using

$$\overline{q}_y\frac{\partial}{\partial t}\eta = -\int d\mathbf{x}'\Big(U(y')\overline{q}_y(y')\delta(\mathbf{x}-\mathbf{x}') + \overline{q}_y(y)G(\mathbf{x}-\mathbf{x}')\overline{q}_y(y')\Big)\frac{\partial}{\partial x'}\eta(\mathbf{x}')$$

For a Fourier mode $\eta = \eta(y,t)\exp(\imath kx)$, we end up with

$$\overline{q}_y\frac{\partial}{\partial t}\eta = -\imath k\int dy'\mathcal{G}(y-y')\eta(y') \quad , \quad \mathcal{G}(y-y') = U(y')\overline{q}_y(y')\delta(y-y') + \overline{q}_y(y)G_k(y-y')\overline{q}_y(y')$$

and the 1-D Green's function $G_k(y-y')$ is found by replacing $\nabla^2$ in (2.2) by $\frac{\partial^2}{\partial y^2} - k^2$. The symmetry implies that $\frac{\partial}{\partial t}\int \overline{q}_y \eta^* \eta = 0$ (Rayleigh's theorem). Also $\frac{\partial}{\partial t}\iint \eta^*(y)\mathcal{G}(y-y')\eta(y') = 0$ (Arnold's thm). Note that we have not explicitly included the sum over layers, but it is implicit.

Since the problem of interest does not satisfy either of the sufficient conditions for stability, we now want to examine the stability more directly. If we look for disturbances of the form $\frac{\partial}{\partial t}\eta = -c\frac{\partial}{\partial x}\eta$ in (2.3), we have

$$U(y)\eta(y) + \int dy' G_k(y-y')\overline{q}_y(y')\eta(y') = c\eta(y) \tag{2.4}$$

Discretizing the integral in $y$ leads to a standard matrix eigenvalue equation. The simplest such comes from replacing $\overline{q}_y$ by delta-functions at the discretization points $y_i$: $\overline{q}_y = q_i\delta(y-y_i)$. Then

$$M_{ij}\eta_j = c\eta_i \quad , \quad M_{ij} = U_i + G_k(y_i - y_j)q_j \tag{2.5}$$

delta=1 delta=0.2 delta=0.2,w=1,v=0 bt delta=0.001,w=1 delta=0.001,w=2 delta=0.001,w=3 delta=0.2,w=1 delta=0.2,w=2 delta=0.2,w=3 delta=0.2,w=3 delta=0.2,deep flow zero

```
% u,y,l:  jet velocities, front locations, layer indices
w=3;v=0.5;
u=[0,1,0,0,v,0];
y=[-w,0,w,-w,0,w];
l=[1,1,1,2,2,2];
% gamma,f:  1/R_d, modal amp.  each mode[columns] & each layer[rows]
del=0.2;
gamma=[0,1];
f=[1,1/sqrt(del);1,-sqrt(del)];
% end of required input
yv=abs(ones(length(y),1)*y-y'*ones(1,length(y)));
% calculate the delta's (PV jumps) from u.  In some cases, g is
% singular and you need to specify delta and perhaps then calculate u
g=jetg(gamma,f,l,yv,0);
delta=-diag(g\u')
ud=diag(u);
% calculate the dispersion relationship
os=[];
k=0.02:0.02:2;
for kk=k
    g=jetg(gamma,f,l,yv,kk);
    mm=ud+g*delta;
    om=eig(mm); os=[os;om'];
end
plot(k,real(os),'+',k,imag(os),'o')
------------- jetg.m ------------
function gs=jetg(gamma,f,l,yv,k)
% function gs=jetg(gamma,f,l,yv,k)
%
% Greens function for jets
%  gamma:  row vector of inv.  def.  rad.;   f:  vertical eigenmode
%  amplitudes in each layer.    l:  layer indices
% yv:  |y_i-y_j|;  k:  wavenumber
gs=zeros(size(yv)*[1;0],size(yv)*[0;1]);
for m=1:size(gamma)*[0;1];
  ff=f(l,m)*f(l,m)';
  ak=sqrt(gamma(m)^2+k^2);
  if ak==0
      gs=gs+ff.*yv/2.0;
  else
      gs=gs-ff.*exp(-ak*yv)/2/ak;
  end
end
```

## 3 —Contour dynamics

Point vortices provide one useful "toy model" for understanding the dynamics. At the next level up, we can consider the PV field as consist of patches of uniform PV. This gives a good method for examining nonlinear effects on the growth of perturbations. (For a smooth $T$ field, it may not be so sensible.)

$$Q = q_i \mathcal{H}\big(y - Y_i(x,t)\big) \quad , \quad Y_i = y_i + \eta_i(x,t)$$

with the mean of $\eta_i$ zero.

The flow can be calculated by integrating around the boundaries of the various patches, as can the streamfunction and the Hamiltonian.

$$H = -\frac{1}{2} h_i q_i q_j \mathcal{I}_i \mathcal{I}'_j G_{ij}(\mathbf{x} - \mathbf{x}') \quad , \quad \mathcal{I}_i = \int_{Y_i}^{\infty} dy \int dx$$

The notation has been simplified here: $h_i$ is for the layer in which interface $i$ is embedded and the indices on $G$ correspond to 1 or 2 depending on the layers holding $i$ and $j$.

We now need the form for the Poisson bracket. We use

$$\delta F = \int dx \int dy \, \frac{\delta F}{\delta q} \big( - q_i \delta(Y_i - y) \nabla \eta_i \big) = \int dx \frac{\delta F}{\delta \eta_i} \delta \eta_i$$

$$\Rightarrow \frac{\delta F}{\delta q} = -\frac{1}{q_i} \frac{\delta F}{\delta \eta_i}$$

The Poisson bracket becomes

$$\begin{aligned} \{A, B\} &= \frac{1}{h_i} \iint dx \, dy \, \left[ \frac{1}{q_i} \frac{\delta A}{\delta \eta_i}, \frac{1}{q_i} \frac{\delta B}{\delta \eta_i} \right] q_i \mathcal{H}(y - Y_i) \\ &= \frac{1}{q_i h_i} \iint dx \, dy \, \frac{\delta A}{\delta \eta_i} \left[ \frac{\delta B}{\delta \eta_i}, \mathcal{H}(y - Y_i) \right] \\ &= \frac{1}{q_i h_i} \int dx \, \frac{\delta A}{\delta \eta_i} \frac{\partial}{\partial x} \frac{\delta B}{\delta \eta_i} \end{aligned}$$

For reference, the bracket for a single curve parameterized by $X(\sigma), Y(\sigma)$ is

$$\{A, B\} = \frac{1}{q_0} \oint d\sigma \left( \frac{Y_\sigma A_X - X_\sigma A_Y}{X_\sigma^2 + Y_\sigma^2} \right) \frac{\partial}{\partial \sigma} \left( \frac{Y_\sigma B_X - X_\sigma B_Y}{X_\sigma^2 + Y_\sigma^2} \right)$$

(subscripts for functional derivatives also). For single-valued contours, using $\sigma = x$, we have $A_X = -\eta_x A_\eta$, $\mathbf{A}_Y = A_\eta$, $X_\sigma = 1$, $Y_\sigma = \eta_x$, and this turns into

$$\{A, B\} = \frac{1}{q_0} \int dx \, A_\eta \frac{\partial}{\partial x} B_\eta$$

as above.

We can find the evolution for the Fourier amplitudes

$$\eta_{i,k} = \frac{1}{2\pi}\int dx\, e^{-\imath kx}\eta_i$$

$$\frac{\partial}{\partial t}\eta_{i,k} = \frac{1}{2\pi h_i q_i}\int dx\, e^{-\imath kx}\frac{\partial}{\partial x}\frac{\delta H}{\delta \eta_i} = \frac{\imath k}{\pi q_i}\int dx\, e^{-\imath kx}\frac{\delta H}{\delta \eta_i}$$

If we use $Yi = y + \eta_k \exp(\imath kx)$, (suppressing the index for the particular contour) then

$$\delta H = \int dx\, \frac{\delta H}{\delta \eta_i}\delta\eta_k e^{\imath kx} = \frac{\partial H}{\partial \eta_n}\delta\eta_n \quad \Rightarrow \quad \int dx\, \frac{\delta H}{\delta \eta_i}e^{-\imath kx} = \frac{\partial H}{\partial \eta_k^*}$$

and we arrive at the dynamical equation for the Fourier amplitudes

$$\frac{\partial}{\partial t}\eta_{i,k} = \frac{\imath k}{2\pi h_i q_i}\frac{\partial H}{\partial \eta_{i,k}^*} \qquad (H - fourier)$$

If we can write an expansion of $H$ in the modal amplitudes $\eta_k$, the equation above will give us information about nonlinear effects on growth. If mode $k$ (denoted by $\eta$) is unstable, we expect to find mode $2k$ (denoted by $\xi$) will also be excited. We will not develop amplitude in mode 0 because of area conservation. To approximate, we expand $H$ around a straight state using

$$H \simeq H_2[\eta_i, \eta_j] + H_3[\eta_i, \eta_j, \eta_m] + H_4[\eta_i, \eta_j, \eta_m, \eta_n]\ldots$$

– a sum of quadratic, cubic, quartic ... functionals of the interface displacements. We truncate to this order. If we keep the $k$ ($\eta$) and $2k$ ($\xi$) terms in the Fourier expansion, we have

$$\begin{aligned} H_2 &= -A_{ij}\eta_i^*\eta_j + \hat{A}_{ij}\xi_i^*\xi_j \\ H_3 &= -B_{ijk}(\eta_i\eta_j\xi_k^* + \eta_i^*\eta_j^*\xi_k) \\ H_4 &= -C_{ijkl}\eta_i^*\eta_j^*\eta_k\eta_l + \ldots \end{aligned}$$

The coefficients are related to the values of the Greens function matrix $\mathcal{G}_{ij}^k = \int dx' \cos(kx')G(x', y_i - y_j, z_i, z_j)$.

The lowest order ($H_2$, $\eta$) gives the same linear equation for the stability problem. We take

$$\eta_i = A(t) e_i$$

As conventional, we use the quasi-equilibrium form in the $\frac{\partial}{\partial t}\xi$ equation to find $\xi \sim \eta\eta^*$. In the fourth order, we end up with

$$\frac{\partial^2}{\partial t^2} A = \sigma^2 A + N A |A|^2$$

We build a jet using 6 contours, three in the upper layer and three in the lower. The jumps are chosen so that the zonal velocity is zero outside the outer contours. We can adjust the deep centerline velocity and the width

1. The long sinuous waves are unstable with $c \sim \imath k^{1/2}$, $\omega \sim \imath k^{3/2}$, even for very thick lower layers (e.g., $\delta = 0.1$).
2. When the deep velocity is prograde and the jet is sufficiently narrow, there is a second, unstable, long wave mode with $c_i \to$ const as $k \to 0$. In the barotropic case, $\upsilon = 1$, Squires' theorem shows that a baroclinic mode of wave number $k$ will be like a barotropic mode with wavenumber $\mu = \sqrt{1+k^2}$. Thus $c_i$ will be non-zero at $k = 0$ for baroclinic perturbations if the barotropic mode with $k = 1$ is unstable. Therefore, for $W < 1.8327$, the very long waves $k \to 0$ will be unstable to a baroclinic mode disturbance, and this mode will have a larger growth rate than the "meandering" mode discussed in 1.
3. We can separate out the varicose modes explicitly by imposing antisymmetry upon the perturbations and working with a reduced $2 \times 2$ matrix Although the geometry of the varicose mode perturbations is similar to Pedlosky's "heton cloud" case, the varicose modes here are stable since the zonal flow vanishes at the $y = L$ interfaces.
4. A narrow jet has a barotropic type of instability even in the limit of an infinitely deep lower layer, but the growth rates decrease as the jet gets wider. In the case of finite lower layer depth, the growth rates level out and correspond to the baroclinic instability of the center fronts.
5. Asymmetric jets have rather similar dispersion characteristics to the symmetric profiles.

6 Generally, we find that nonlinearity *enhances* the instability for widths less than 2.3 (in units of the deformation radius; recall that the "width" $W$ is the distance to the zero in the velocity). For wider jets, the nonlinearity leads to equilibration as occurs for the baroclinic front Flierl (1988). For fairly barotropic jets, the nonlinearity is again destabilizing; in this case, numerical studies of the barotropic jet suggests that vortex street formation will occur.

The case of the "heton cloud" with just one front in each layer is especially interesting. The linear matrix when $h_1 = h_2 = 0.5$ gives Green's functions

$$G_{11} = G_{22} = -\frac{1}{4k} e^{-k|y-y'|} - \frac{1}{4K} e^{-K|y-y'|} \quad , \quad G_{12} = G_{21} = -\frac{1}{4k} e^{-k|y-y'|} + \frac{1}{4K} e^{-K|y-y'|}$$

with $K^2 = k^2 + \gamma^2$. We take

$$\overline{u} = \pm U e{-}\gamma|y|$$

corresponding to $q_i = \pm 2U\gamma$. The resulting matrix

$$\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \frac{1}{4kK} \begin{pmatrix} -k-K & k-K \\ k-K & -k-K \end{pmatrix} \begin{pmatrix} 2\gamma U & 0 \\ 0 & -2\gamma U \end{pmatrix}$$

leading to a dispersion relation

$$c = \pm U \sqrt{\frac{(k-\gamma)(K-\gamma)}{kK}}$$

which becomes unstable when $k < \gamma$. Short waves are trapped near the front in either the upper or lower layer, depending on the branch of the dispersion relation, and propagate non-dispersively with the speed at the front in the appropriate layer. Long waves are unstable with a growth rate $\sigma = \Im(\omega) \propto k^{3/2}$, in agreement with the prediction of thin jet theory (Flierl and Robinson, 1984, Howard and Drazin, 19xx). We would expect the thin jet theory to be applicable in this case since the momentum transport of the basic jet is finite.

The nonlinear results indicate this is a supercritical bifurcation, although the nonlinear term changes sign when $k/\gamma$ becomes smaller. When the perturbation is no longer bounded $N > 0$, the symmetry of the system results in a wave which grows to very large amplitude until the resulting N-S jets interact and break up the feature.

To investigate, this, we used a standard QG model with $k = 1$ as the longest wave in the domain. We also calculated steady states using simulated annealing.

**4 —Nonlinear coefficient**

$$H = -\frac{1}{2} q_i q_j \int_{A_i} d\mathbf{x} \int_{A_j} d\mathbf{x}'\, G(\rho) \quad , \quad \rho = |\mathbf{x} - \mathbf{x}'|$$

Let

$$\nabla^2 \Phi(\rho) = G(\rho)$$

$$\begin{aligned}
H &= -\frac{1}{2} q_i q_j \int_{A_i} d\mathbf{x} \int_{A_j} d\mathbf{x}' \nabla'^2 \Phi \\
&= -\frac{1}{2} q_i q_j \int_{A_i} d\mathbf{x} \oint_{A_j} ds' \nabla' \Phi \cdot \hat{\mathbf{n}}' \\
&= -\frac{1}{2} q_i q_j \int_{A_i} d\mathbf{x} \oint_{A_j} ds' \Phi_\rho \nabla' \rho \cdot \hat{\mathbf{n}}' \\
&= \frac{1}{2} q_i q_j \int_{A_i} d\mathbf{x} \oint_{A_j} ds' \Phi_\rho \nabla \cdot (\rho \hat{\mathbf{n}}') \quad \text{since} \quad \nabla' \rho = -\nabla \rho \\
&= \frac{1}{2} q_i q_j \int_{A_i} d\mathbf{x} \oint_{A_j} ds' \nabla \cdot (\Phi \hat{\mathbf{n}}') \\
&= \frac{1}{2} q_i q_j \oint_{A_i} ds \oint_{A_j} ds'\, \Phi\, \hat{\mathbf{n}} \cdot \hat{\mathbf{n}}'
\end{aligned}$$

For contours $Y_i = y_i + \eta_i(x)$

$$\hat{\mathbf{n}} = \left( \frac{1}{\sqrt{1+\eta_x^2}}, \frac{-\eta_x}{\sqrt{1+\eta_x^2}} \right)$$

$$\rho^2 = (x - x')^2 + \big(y_i + \eta_i(x) - y_j - \eta_j(x')\big)^2$$

Taylor expand in $\eta$ to some order ($\eta^4$) and Fourier expand ($k$, $2k$).

Turn into equations for $\frac{d}{dt}\eta_i(k)$ and $\eta_i(2k)$ and make the usual amplitude expansion neat the critical point. Or use

$$\hat{\eta}_i(k) = a(t)e_i + \frac{1}{\imath k}\frac{\partial}{\partial t} a f_i \quad , \quad \hat{\eta}_i(2k) = a^2 g_i$$

in $A = H + c_r P$ and set $\frac{\partial}{\partial t}A = 0$. Here $e_i$ is the eigenvector at the critical point $k_0$, $f_i$ is the correction when $k$ is slightly less than $k_0$, and $g_i$ is the forced $2k$ vector.

$$\frac{\partial^2}{\partial t^2} a = \sigma^2 a + N a|a|^2$$

### *4.1 — Numerical results*

For the heton cloud, $max(Y_1)$ is:

heton k=1/1.05

k=1/1.1

sa

| $\gamma$ | 0.025 | 0.05 |
|---|---|---|
| 0.98 | 0.097266 | 1.8738 |
| 1.02 | 0.73157 | 0.75829 |
| 1.04 | 1.2710 | 1.2812 |
| 1.05 | 1.5990 | 1.6049 |
| 1.06 | 1.9995 | 2.0067 |
| 1.08 | 3.4278 | 3.4361 |
| 1.1 | – | – |

### *4.2 — Conclusions: numerical*

1) Nonlinearly stable regions exist but seem to be very narrow.
2) Most disturbances will break via eddy formation for the more baroclinic jet or street formation for the more barotropic cases.

*4.3 — Pulses*

Although almost all stability computations are for zonal flow and normal modes or at best wavenumber $k$, most atmospheric and oceanic disturbances are much more localized in space and time.

If we consider the evolution of a localized initial disturbance in 1D, using the method of Farrell (19xx), who shows that the perturbation behaves like

$$\eta(x,t) = \int dk A(k) e^{\imath kx - \imath\Omega(k)t}$$

$$\eta(x,t) = \lim_{t\to\infty} \int dk A(k) e^{\imath(kU - \Omega(k))t}$$

The steps are:

- Deform the integration contour in the complex $k$ plane to pass through the saddle point $k_s$ such that $U = d\Omega/dk$. Then

  $$\eta(x,t) \to A(k_s) e^{\imath(k_s U - \Omega(k_s))t} \int dk' \exp\left(-\imath t \frac{d^2\Omega}{dk^2}\Big|_{k_s} k'^2\right)$$

  The envelope will be growing at a rate

  $$\sigma = \Im\big(\Omega(k_s) - Uk_s\big)$$

- The saddle point wavenumber is determined by

  $$\frac{d}{dk}(\omega - k\frac{x}{t}) = 0$$

  in the complex $k$ plane. The required derivative

  $$\frac{d\omega}{dk} = c + k\frac{dc}{dk}$$

  can be found using the matrix eigenvalue problem (4.5). If we differentiate this with respect to $k$, we find

  $$\left[\frac{d}{dk}M_{ij}\right]\eta_j + M_{ij}\left[\frac{d}{dk}\eta_j\right] = \left[\frac{d}{dk}c\right]\eta_i + c\left[\frac{d}{dk}\eta_i\right]$$

  Multiplying by the left eigenvector $\eta_i^+$ of $M_{ij}$ gives

  $$[\frac{d}{dk}c\,] = \eta_i^+\left[\frac{d}{dk}M_{ij}\right]\eta_j/\eta_i^+\eta_j$$

  We then search out the zeros of $d\omega_i/dk$ in the complex $k$ space beginning with the one known to be on the real $k$ axis, The envelope shape as a function of $x/t = d\omega_r/dk$

for the two $\delta$ values we've been using. For both values of $\delta$, absolute instability can occur, though the most rapidly growing part of the envelope moves downstream at a speed of about 12% of the maximum upper layer jet velocity.

examples

The pulse asymptotics is a bit more complex for the six front jets, since there may be several unstable modes. However, the picture which emerges is fairly simple: the jets with prograde lower layer flow are convectively unstable, with the peak of the envelope growing (as it must) at the maximum temporal growth rate and moving downstream at about 23% of the maximum jet speed. When the lower layer flows are retrograde at the center, the jet has an absolute instability, though the peak of the envelope still moves (more slowly) downstream.

*4.4 — Continuum modes*

One criticism of contour dynamics models for jets is that they lack the continuous spectrum. To compensate for this lack, however, they have more neutral normal modes. As the number of contours becomes large, the set of the frequencies of these modes becomes dense. If we consider an initial condition with many of these modes excited and appropriately out of phase, they can very gradually come into phase and then shift out again. The recurrence (or near recurrence) time can be very long. Therefore transient growth and decay of an initial disturbance is actually similar: there is a long period with the amplitude proportional to $t^{-2}$ as obtained for the continuous jet by Brown and Stewartson (19xx). Notice that a large number of contours (xx) is required to give the long recurrence time.

*4.5 — Zero lower layer flow*

The basic state we assume in this section also has a continuous PV distribution, with a single PV front in the upper layer and no basic state flow whatsoever in the lower layer. This basic state can be approximated by a large number of small amplitude fronts in the lower layer to mimic the no-flow condition there. We have also derived the dispersion relation from a Frobenius expansion of the integral eigenvalue problem (4.4). The results are identical.

The growth rates show a single unstable mode corresponding to a baroclinic instability. The maximum growth rate occurs at a scale x.x and decreases as the lower layer becomes deeper, being proportional to $\delta$. These results are the same as the standard two–layer baroclinic instability problem for a $y$–independent zonal flow. The waves on the upper layer PV contour with a positive northward jump reinforce the perturbations in the lower layer negative PV gradient regions.

## 5 —Ring formation

The heton model and the upper-layer front both have the essential elements for ring formation. In essence, the upper layer positive PV jump interacts with the negative deep PV jump or gradient so that the perturbations amplify each other. The propagation speeds need to match so that the phase offset is maintained. As the bump elongates, the anomaly begins to move the south-eastern side backwards until it pinches off. The meander needs to be long enough to allow this to occur while retaining the water properties within the ring. sketch 0-20 30-50 60-80 90-100

## 6 —Diagnostics of baroclinic amplification

The equations for the upper layer

$$\frac{\partial}{\partial t} q_1 = [q_1, \psi_1] = \left[ q_1, \int G_{11}(\mathbf{x} - \mathbf{x}') q_1(\mathbf{x}') \right] + \left[ q_1, \int G_{12}(\mathbf{x} - \mathbf{x}') q_2(\mathbf{x}') \right]$$

show that we can calculate separately the motion tendency from vortex interactions within the same layer and from the other layer. This "piecewise inversion" allows us to diagnose baroclinic amplification: the velocities induced on the central upper layer PV contour by the lower layer flows tend to elongate it, and likewise the upper layer influence on the lower layer is amplifying the displacement. from thin jet from thin jet from westrex

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