

Detailed formulation of SCALE-DG

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1 Introduction

1.1 What is SCALE-DG?

FE-project provides a library and sample programs for the discontinuous Galerkin methods (DGM). Furthermore, we provide an atmospheric model with a regional/global dynamical core based on DGM, SCALE-DG. In FE-Project, we use scalable Computing for Advanced Library and Environment (SCALE), which is a basic software library of weather and climate models of the earth and planets intended for widespread use.

The general references of SCALE-DG are Kawai and Tomita (2023, 2025). If SCALE-DG is used in your studies, please cite two our papers in addition to the reference papers of SCALE library (Nishizawa et al., 2015; Sato et al., 2015).

2 Governing equations

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2.1 Coordinate system

To describe the governing equations for both regional and global dynamical cores, a non-orthogonal curvilinear horizontal coordinate (ξ, η) and a general vertical coordinate ζ are introduced, following Li et al. (2020). For the horizontal coordinate transformation, the contravariant form of the metric tensor is represented by G_h^{ij} for $i, j = 1, 2$. A three-dimensional metric tensor with the horizontal coordinate transformation is defined as

$$G^{ij} = \begin{pmatrix} G_h^{11} & G_h^{12} & 0 \\ G_h^{21} & G_h^{22} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (2.1)$$

The horizontal Jacobian is defined as $\sqrt{G_h} = |G_h^{ij}|^{-\frac{1}{2}}$. The Christoffel symbol of the second kind Γ_{ml}^i , which means the spatial variation of the basis vector, is represented as

$$\Gamma_{ml}^i = \frac{1}{2} G^{im} \left(\frac{\partial G_{jm}}{\partial x^k} + \frac{\partial G_{km}}{\partial x^j} + \frac{\partial G_{jk}}{\partial x^m} \right) \quad (2.2)$$

For the vertical coordinate transformation, the metric tensor is defined as $G_v^{13} = \partial\zeta/\partial\xi$, $G_v^{23} = \partial\zeta/\partial\eta$ and the vertical Jacobian is defined as $\sqrt{G_v} = \partial z/\partial\zeta$. The vertical velocity in the transformed vertical coordinate can be written using contravariant components of wind vector (u^ξ, u^η, u^ζ) as

$$\tilde{u}^\zeta \equiv \frac{d\zeta}{dt} = \frac{1}{\sqrt{G_v}} \left(u^\zeta + \sqrt{G_v} G_v^{13} u^\xi + \sqrt{G_v} G_v^{23} u^\eta \right). \quad (2.3)$$

The final Jacobian composed of horizontal and vertical coordinate transformations can be represented as $\sqrt{G} = \sqrt{G_h} \sqrt{G_v}$. Hereafter, to briefly describe the formulations, the coordinate variables are sometimes expressed using $(\xi^1, \xi^2, \xi^3) = (\xi, \eta, \zeta)$. In addition, the Einstein summation notation will be applied for repeated indices when representing the geometric relations.

2.1.1 Horizontal coordinates used in regional model

For horizontal coordinates in our regional model, the horizontal Cartesian coordinates (x, y) is simply adopted although we will introduce map projections in the near future.

Then,

$$G_h^{ij} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \sqrt{G_h} = 1, \quad (2.4)$$

The Christoffel symbol of the second kind Γ_{ml}^i is represented as

$$\Gamma_{ml}^1 = 0, \quad \Gamma_{ml}^2 = 0, \quad \Gamma_{ml}^3 = 0, \quad (2.5)$$

where $m, l = 1, 2, 3$.

The components of angular velocity vector included in the Coriolis terms are given as

$$\begin{aligned} \Omega^1 &= 0, \quad \Omega^2 = 0, \\ \Omega^3 &= f_0 + \beta y. \end{aligned} \quad (2.6)$$

Here, $f_0 = 2\Omega \sin \theta_0$, $\beta = 2\Omega \cos \theta_0$ where ω is the angular velocity of the planet and θ_0 is the reference latitude.

2.1.2 Horizontal coordinates used in global model

For horizontal coordinates in our global model, we adopt an equiangular gnomonic cubed-sphere projection (Sadourny, 1972; Ronchi et al., 1996) to map a cube onto a sphere. Compared to a conformal projection (Rančić et al., 1996), we prefer this projection to generate more uniform grids in high spatial resolutions, although the non-orthogonal basis need to be treated. In each panel of the cube, a local coordinate using the central angles (α, β) ($\in [-\pi/4, \pi/4]$) was introduced and related to the horizontal coordinates (ξ, η) by $\xi = \alpha, \eta = \beta$. Based on the derivation with the coordinate transformation in previous studies (e.g., Nair et al., 2005; Ullrich and Jablonowski, 2012b; Li et al., 2020), the horizontal contravariant metric tensor and the horizontal Jacobian for the equiangular gnomonic cubed-sphere projection can be written as, respectively,

$$G_c^{ij} = \frac{\delta^2}{r^2(1+X^2)(1+Y^2)} \begin{pmatrix} 1+Y^2 & XY \\ XY & 1+X^2 \end{pmatrix}, \quad \sqrt{G_c} = \frac{r^2(1+X^2)(1+Y^2)}{\delta^3}, \quad (2.7)$$

where $X = \tan \alpha$, $Y = \tan \beta$, $\delta = \sqrt{1+X^2+Y^2}$, and r is the radial coordinate. The Christoffel symbol of the second kind Γ_{ml}^i is represented as

$$\begin{aligned} \Gamma_{ml}^1 &= \begin{pmatrix} \frac{2XY^2}{\delta^2} & \frac{-Y(1+Y^2)}{\delta^2} & \frac{\delta_S}{r} \\ \frac{-Y(1+Y^2)}{\delta^2} & 0 & 0 \\ \frac{\delta_S}{r} & 0 & 0 \end{pmatrix}, \\ \Gamma_{ml}^2 &= \begin{pmatrix} 0 & \frac{-X(1+X^2)}{\delta^2} & 0 \\ \frac{-X(1+X^2)}{\delta^2} & \frac{2X^2Y}{\delta^2} & \frac{\delta_S}{r} \\ 0 & \frac{\delta_S}{r} & 0 \end{pmatrix}, \\ \Gamma_{ml}^3 &= \delta_S \frac{r(1+X^2)(1+Y^2)}{\delta^4} \begin{pmatrix} -(1+X^2) & XY & 0 \\ XY & -(1+Y^2) & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \quad (2.8)$$

where δ_S is a switch for shallow atmosphere approximation.

The components of angular velocity vector included in the Coriolis terms are given as

$$\begin{aligned} \Omega^1 &= 0, \quad \Omega^2 = \delta_S \frac{\omega \delta}{r(1+Y^2)}, \quad \Omega^3 = \omega \frac{Y}{\delta}, \quad \text{for the equatorial panels,} \\ \Omega^1 &= -\delta_S \frac{s\omega X \delta}{r(1+X^2)}, \quad \Omega^2 = -\delta_S \frac{s\omega Y \delta}{r(1+Y^2)}, \quad \Omega^3 = \frac{s\omega}{\delta}, \quad \text{for the polar panels,} \end{aligned} \quad (2.9)$$

where ω is the angular velocity of the planet and an index s has a value of 1 and -1 for the Northern and Southern polar panels, respectively.

2.1.3 Vertical coordinates

To treat the topography, we adopt the traditional terrain-following coordinate (Phillips, 1957; Gal-Chen and Somerville, 1975) as a general vertical coordinate. The vertical coordinate transformation can be expressed as

$$\zeta = z_T \frac{z - h}{z_T - h}, \quad (2.10)$$

where z is the height coordinate, z_T is the top height of computational domain (we assume it is a constant value), and h is the surface height. The corresponding Jacobian and metric tensor can be represented as

$$\sqrt{G_v} = 1 - \frac{h}{z_T}, \quad \sqrt{G_v} G_v^{13} = \left(\frac{\zeta}{z_T} - 1 \right) \frac{\partial h}{\partial \xi}, \quad \sqrt{G_v} G_v^{23} = \left(\frac{\zeta}{z_T} - 1 \right) \frac{\partial h}{\partial \eta}, \quad (2.11)$$

respectively.

2.2 Governing equations for atmospheric dynamical core

As the governing equations, we adopt the three-dimensional, fully compressible nonhydrostatic equations based on the flux form (e.g., Ullrich and Jablonowski, 2012b). The compact form of the governing equations can be written as

$$\begin{aligned} \frac{\partial \mathbf{q}}{\partial t} + \frac{\partial [\mathbf{f}(\mathbf{q}) + \mathbf{f}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q})]}{\partial \xi} + \frac{\partial [\mathbf{g}(\mathbf{q}) + \mathbf{g}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q})]}{\partial \eta} + \frac{\partial [\mathbf{h}(\mathbf{q}) + \mathbf{h}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q})]}{\partial \zeta} \\ = \mathbf{S}(\mathbf{q}) + \mathbf{S}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q}), \end{aligned} \quad (2.12)$$

Here, $\mathbf{q}$ is the solution vector defined as

$$\mathbf{q} = \left(\sqrt{G} \rho', \sqrt{G} \rho u^\xi, \sqrt{G} \rho u^\eta, \sqrt{G} \rho u^\zeta, \sqrt{G} (\rho \theta)', \rho q_* \right)^T, \quad (2.13)$$

where ρ, θ denote the density and potential temperature defined later, respectively. q_* represents the mass concentration of each material such as water vapor (q_v), liquid water (q_l), and solid water (q_s). The mass concentrations should be the relation as

$$q_d + \sum_{*=v,l,s} q_* = 1, \quad (2.14)$$

where q_d is the mass concentration of dry air. To accurately treat nearly balanced flows, the density ρ and pressure p (thus $\rho\theta$) are decomposed as $q(\xi, \eta, \zeta, t) = q_{\text{hyd}}(\xi, \eta, \zeta) + q'(\xi, \eta, \zeta, t)$, where q_{hyd} denotes a variable satisfying the hydrostatic balance and q' denotes the deviation.

In Eq. (2.12), $\mathbf{f}(\mathbf{q})$, $\mathbf{g}(\mathbf{q})$, and $\mathbf{h}(\mathbf{q})$ are inviscid fluxes in the ξ , η , and ζ directions, respectively. The horizontal inviscid fluxes are represented as

$$\mathbf{f}(\mathbf{q}) = \begin{pmatrix} \sqrt{G}\rho u^\xi \\ \sqrt{G}(\rho u^\xi u^\xi + G_h^{11}p') \\ \sqrt{G}(\rho u^\eta u^\xi + G_h^{21}p') \\ \sqrt{G}\rho u^\zeta u^\xi \\ \sqrt{G}\rho\theta u^\xi \\ \sqrt{G}\rho q_* u^\xi \end{pmatrix}, \quad \mathbf{g}(\mathbf{q}) = \begin{pmatrix} \sqrt{G}\rho u^\eta \\ \sqrt{G}(\rho u^\xi u^\eta + G_h^{12}p') \\ \sqrt{G}(\rho u^\eta u^\eta + G_h^{22}p') \\ \sqrt{G}\rho u^\zeta u^\eta \\ \sqrt{G}\rho\theta u^\eta \\ \sqrt{G}\rho q_* u^\eta \end{pmatrix}, \quad (2.15)$$

and the vertical inviscid fluxes are represented as

$$\mathbf{h}(\mathbf{q}) = \begin{pmatrix} \sqrt{G}\rho \tilde{u}^\zeta \\ \sqrt{G}[\rho u^\xi \tilde{u}^\zeta + (G_v^{13}G_h^{11} + G_v^{23}G_h^{12})p'] \\ \sqrt{G}[\rho u^\eta \tilde{u}^\zeta + (G_v^{13}G_h^{21} + G_v^{23}G_h^{22})p'] \\ \sqrt{G}\rho u^\zeta \tilde{u}^\zeta + \sqrt{G}p' \\ \sqrt{G}\rho\theta \tilde{u}^\zeta \\ \sqrt{G}\rho q_* \tilde{u}^\zeta \end{pmatrix}. \quad (2.16)$$

Furthermore, $\mathbf{S}(\mathbf{q})$ in Eq. (2.12) represents the source terms as

$$\mathbf{S}(\mathbf{q}) = \begin{pmatrix} \sqrt{G}S_{\rho, \text{phy}} \\ \sqrt{G}(F_H^1 + F_M^1 + F_C^1) + \sqrt{G}S_{\rho u^\xi, \text{phy}} \\ \sqrt{G}(F_H^2 + F_M^2 + F_C^2) + \sqrt{G}S_{\rho u^\eta, \text{phy}} \\ \sqrt{G}(F_{\text{buo}} + F_C^3) + \sqrt{G}S_{\rho u^\zeta, \text{phy}} \\ \sqrt{G}S_{\rho\theta, \text{phy}} \\ \sqrt{G}S_{\rho q_*, \text{phy}} \end{pmatrix}, \quad (2.17)$$

where F_H^i for $i = 1, 2$ are the horizontal pressure gradient terms with hydrostatic balance and can be expressed as

$$F_H^i = -\frac{G_h^{im'}}{\sqrt{G_v}} \left[\frac{\partial(\sqrt{G_v}p_{\text{hyd}})}{\partial \xi^{m'}} + \frac{\partial(G_v^{m'3}\sqrt{G_v}p_{\text{hyd}})}{\partial \xi^3} \right]. \quad (2.18)$$

Here, note that $m' = 1, 2$. F_M^i is the source terms due to the horizontal curvilinear coordinate as

$$F_M^i = -\Gamma_{ml}^i(\rho u^m u^l + G^{ml}p'), \quad (2.19)$$

where m, l take values of 1, 2, 3. F_C^i is the Coriolis terms as

$$F_C^i = -G^{im}\epsilon_{jml}2\Omega^m\rho u^l, \quad (2.20)$$

where ϵ_{jkl} is the three rank Levi-Civita tensor and Ω^m are the components of angular velocity vector. F_{buo} is the buoyancy term as

$$F_{\text{buo}} = -\rho' \left(\frac{a}{r} \right)^2 g, \quad (2.21)$$

where r is the radial coordinate, a is the planetary radius, and g is the standard gravitational acceleration. In Eq. (2.12), the terms with subscript SGS are associated with a turbulent model; $\mathbf{f}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q})$, $\mathbf{g}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q})$, and $\mathbf{h}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q})$ are the parameterized eddy fluxes while $\mathbf{S}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q})$ are the source terms with the curvilinear coordinates. The terms associated with the turbulent model are detailed in Sect. 4.1. On the other hand, $\mathbf{S}_{*,\text{phys}}$ represents the terms with physical processes such as cloud, radiation, and surface processes.

To consider the moist thermodynamics, we introduce a potential temperature defined as

$$\theta = T \left(\frac{P_0}{p} \right)^{R^*/C_p^*}, \quad (2.22)$$

where T is the temperature, P_0 is a constant pressure. The gas constant R^* and the specific heat at constant volume C_p^* are defined as

$$C_p^* = \sum_{*=d,v,l,s} q_* C_{p,*}, \quad R^* = \sum_{*=d,v,l,s} q_* R_*. \quad (2.23)$$

To close the equation systems in Eq. (2.12), the pressure p is calculated using

$$p = P_0 \left(\frac{R^*}{P_0} \rho \theta \right)^{\frac{C_p^*}{C_v^*}}, \quad (2.24)$$

where $C_v^* = C_p^* - R^*$. In terms of C_p^* and R^* , a diabatic heating contribution $H_{\rho\theta}$ included in $\mathbf{S}_{\rho\theta,\text{phys}}$ can be written as

$$H_{\rho\theta} = \frac{1}{C_p^*} \left(\frac{P_0}{p} \right)^{\frac{R^*}{C_p^*}} \left\{ Q - \frac{p}{R^*} \left[\dot{C}_p^* + \ln \left(\frac{p}{P_0} \right) \left(\frac{C_v^*}{C_p^*} \dot{C}_p^* - \dot{C}_v^* \right) \right] \right\}, \quad (2.25)$$

where Q is the diabatic heating with the unit J/(m³s). The second term in the braces accounts for changes in $\rho\theta$ arising from temporal variations in the thermodynamic properties of the atmospheric mixture, rather than from diabatic heating itself.

When the traditional approximation is applied to the governing equations in global model, δ_S should be set to zero.

3 Discretization of dynamics

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3.1 Spatial discretization

We perform the spatial discretization for Eq. (2.12) based on a nodal DGM (e.g., Hesthaven and Warburton, 2007). The three-dimensional computational domain Ω is divided using non-overlapping hexahedral elements. To relate the coordinates (ξ^1, ξ^2, ξ^3) with the local coordinates $\tilde{\mathbf{x}} \equiv (\tilde{x}^1, \tilde{x}^2, \tilde{x}^3)$ in a reference element Ω_e , we adopted a linear mapping defined as

$$\tilde{x}^i = 2 \frac{\xi^i - \xi_e^i}{h_e^i}, \quad (3.1)$$

where ξ_e^i and h_e^i represent the center position and width of the element, respectively, in the ξ^i -direction.

Using the tensor-product of one-dimensional Lagrange polynomials

$$l_{\mathbf{m}}(\tilde{\mathbf{x}}) = l_{m_1}(\tilde{x}^1) l_{m_2}(\tilde{x}^2) l_{m_3}(\tilde{x}^3), \quad (3.2)$$

a local approximated solution within each element Ω_e can be represented as

$$\mathbf{q}^e|_{\Omega_e}(\tilde{\mathbf{x}}, t) = \sum_{m_1=1}^{p+1} \sum_{m_2=1}^{p+1} \sum_{m_3=1}^{p+1} \mathbf{Q}_{m_1, m_2, m_3}^e(t) l_{m_1}(\tilde{x}^1) l_{m_2}(\tilde{x}^2) l_{m_3}(\tilde{x}^3), \quad (3.3)$$

In Eq. (3.3), the coefficients Q_{m_1, m_2, m_3}^e are the unknown degrees of freedom (DOF) and p is the polynomial order. In this study, the Legendre–Gauss–Lobatto (LGL) points were used for interpolation and integration nodes.

Semi-discretized equations

By applying the Galerkin approximation to Eq. (2.12), a strong form of the semi-discretized equations can be obtained as

$$\begin{aligned} \frac{D}{Dt} \int_{\Omega_e} \mathbf{q}^e(\tilde{\mathbf{x}}, t) l_{\mathbf{m}}(\tilde{\mathbf{x}}) J^E d\tilde{\mathbf{x}} = & - \sum_{j=1}^3 \int_{\Omega_e} \frac{\partial \mathbf{F}_j(\mathbf{q}^e, \mathbf{G})}{\partial \xi^j} l_{\mathbf{m}}(\tilde{\mathbf{x}}) J^E d\tilde{\mathbf{x}} \\ & - \int_{\partial\Omega_e} \left[\hat{\mathbf{F}}(\mathbf{q}^e, \mathbf{G}) - \mathbf{F}(\mathbf{q}^e, \mathbf{G}) \right] \cdot \mathbf{n} l_{\mathbf{m}}(\tilde{\mathbf{x}}) J^{\partial E} dS \\ & + \int_{\Omega_e} [\mathbf{S}(\mathbf{q}^e) + \mathbf{S}_{\text{SGS}}(\mathbf{q}^e, \mathbf{G})] l_{\mathbf{m}}(\tilde{\mathbf{x}}) J^E d\tilde{\mathbf{x}}, \end{aligned} \quad (3.4)$$

where $(\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_3) = (\mathbf{f} + \mathbf{f}_{\text{SGS}}, \mathbf{g} + \mathbf{g}_{\text{SGS}}, \mathbf{h} + \mathbf{h}_{\text{SGS}})$ is the flux vector tensor, $\hat{\mathbf{F}}$ is the numerical flux at the element boundary $\partial\Omega_e$, and $\mathbf{n}$ is the outward unit vector normal to $\partial\Omega_e$; In the volume and surface integrals, J^E and $J^{\partial E}$ represent the transformation Jacobian with the general curvilinear coordinates and local coordinates within each element. Note that, because of the linear mapping in Eq. (3.1), the associated geometric factors such as J^E and $J^{\partial E}$ have constant values when the volume and surface integrals are calculated.

For turbulent models, we need to evaluate the eddy viscous flux tensor and diffusion flux, which include a few gradient terms with quantities such as $\chi = (u^\xi, u^\eta, u^\zeta, \theta, q_{v,l,s})$, denoted by $\mathbf{G} = (\partial\chi/\partial\xi^1, \partial\chi/\partial\xi^2, \partial\chi/\partial\xi^3)$ in Eq. (3.4). The gradient discretization in the ξ^j -direction is given by

$$\begin{aligned} \int_{\Omega_e} \rho \mathbf{G}_j l_m(\tilde{\mathbf{x}}) J^E d\tilde{\mathbf{x}} &= \int_{\Omega_e} \left[\frac{\partial \rho^e \chi^e}{\partial \xi^j} - \chi^e \left(\frac{\partial \rho}{\partial \xi^j} \right)^e \right] l_m(\tilde{\mathbf{x}}) J^E d\tilde{\mathbf{x}} \\ &+ \int_{\partial\Omega_e} (\widehat{\rho\chi} - \rho^e \chi^e) \mathbf{n}_{\tilde{x}^j} \cdot \mathbf{n} l_m(\tilde{\mathbf{x}}) J^{\partial E} dS, \end{aligned} \quad (3.5)$$

where $\mathbf{n}_{\tilde{x}^j}$ is the unit vector in the $\tilde{x}^j$ -direction and the density gradient is calculated by

$$\int_{\Omega_e} \left(\frac{\partial \rho}{\partial \xi^j} \right)^e l_m(\tilde{\mathbf{x}}) J^E d\tilde{\mathbf{x}} = \int_{\Omega_e} \frac{\partial \rho^e}{\partial \xi^j} l_m(\tilde{\mathbf{x}}) J^E d\tilde{\mathbf{x}} + \int_{\partial\Omega_e} (\hat{\rho} - \rho^e) \mathbf{n}_{\tilde{x}^j} \cdot \mathbf{n} l_m(\tilde{\mathbf{x}}) J^{\partial E} dS. \quad (3.6)$$

Numerical flux

For the numerical flux of the inviscid terms, the Rusanov flux (Rusanov, 1961) is used as a simple choice of the approximated Riemann solvers. Its numerical dissipation is provided based on the maximum absolute eigenvalue of the Jacobian matrix at the left and right sides of the element boundary. The Rusanov flux is written as

$$\hat{\mathbf{F}}_{\text{invis}} = \frac{1}{2} \{ [\mathbf{F}_{\text{invis}}(\mathbf{q}^+) + \mathbf{F}_{\text{invis}}(\mathbf{q}^-)] \cdot \mathbf{n} - \lambda_{\max} [\mathbf{q}^+ - \mathbf{q}^-] \}, \quad (3.7)$$

where $\lambda_{\max}$ is the maximum of the absolute value of eigenvalues of the flux Jacobian in the direction $\mathbf{n}$, and $\mathbf{q}^-$ and $\mathbf{q}^+$ represent the interior and exterior values at $\partial\Omega_e$. Previous studies (e.g., Li et al., 2020) formulated the Rusanov flux taken into account the horizontal and vertical coordinate transformations. Based on their works, at the element boundaries in the horizontal directions (ξ and η), $\lambda_{\max}$ can be represented as

$$\lambda_{\max, \xi} = |u^\xi| + \sqrt{G_h^{11} c_s}, \quad \lambda_{\max, \eta} = |u^\eta| + \sqrt{G_h^{22} c_s}, \quad (3.8)$$

where $c_s = [(C_p/C_v)RT]^{1/2}$ is the speed of sound wave. For the vertical direction ζ , $\lambda_{\max}$ can be represented as

$$\lambda_{\max, \zeta} = |\tilde{u}^\zeta| + \left[1/\sqrt{G_v} + G_v^{13} G_X + G_v^{23} G_Y \right]^{1/2} c_s, \quad (3.9)$$

where $G_X = G_v^{13} G_h^{11} + G_v^{23} G_h^{12}$ and $G_Y = G_v^{13} G_h^{21} + G_v^{23} G_h^{22}$.

We adopt the central flux as the numerical flux of the gradient $\mathbf{G}$ and the SGS fluxes $(\mathbf{f}_{\text{SGS}}, \mathbf{g}_{\text{SGS}}, \mathbf{h}_{\text{SGS}})$ with the turbulent model.

Matrix form of semi-discretized equation

When the same nodes are used for interpolation and integration (i.e., collocation), a matrix form of Eqs. (3.4) and (3.5) can be obtained as

$$\begin{aligned} \frac{D\mathbf{q}^e}{Dt} = & - \sum_{j=1}^3 d_j D_{\tilde{x}^j} \mathbf{F}_j(\mathbf{q}^e, \mathbf{G}) - \sum_{f=1}^6 s_{\partial\Omega_{e,f}} L_{\partial\Omega_{e,f}} \left[\hat{\mathbf{F}}(\mathbf{q}^e, \mathbf{G}) - \mathbf{F}(\mathbf{q}^e, \mathbf{G}) \right] \cdot \mathbf{n} \\ & + \mathbf{S}(\mathbf{q}^e) + \mathbf{S}_{\text{SGS}}(\mathbf{q}^e, \mathbf{G}), \end{aligned} \quad (3.10)$$

$$\rho \mathbf{G}_j = d_j D_{\tilde{x}^j} (\rho^e \boldsymbol{\chi}^e) - \boldsymbol{\chi}^e \left(\frac{\partial \rho}{\partial \xi^j} \right)^e + \sum_{f'=1}^2 s_{\partial\Omega_{e,f'}} L_{\partial\Omega_{e,f'}} (\widehat{\rho \boldsymbol{\chi}} - \rho^e \boldsymbol{\chi}^e) \mathbf{n}_{\tilde{x}^j} \cdot \mathbf{n}, \quad (3.11)$$

where $D_{\tilde{x}^j}$ represents the differential matrix for the $\tilde{x}^j$ -direction; $L_{\partial\Omega_{e,f}}$ represents the lifting matrix with the surface integral for the f -th element surface, and $L_{\partial\Omega_{e,f'}}$ represents the same for the f' -th element surface in the gradient operator for the $\tilde{x}^j$ -direction. The components of these matrices are given as

$$(D_{\tilde{x}^j})_{\mathbf{m}, \mathbf{m}'} = M^{-1} \int_{\Omega_e} l_{\mathbf{m}} \frac{\partial l_{\mathbf{m}'}}{\partial \tilde{x}^j} d\tilde{\mathbf{x}}, \quad (L_{\partial\Omega_{e,j}})_{\mathbf{m}, \mathbf{m}'} = M^{-1} \int_{\partial\Omega_{e,j}} l_{\mathbf{m}} l_{\mathbf{m}'}^{\partial\Omega_{e,j}} dS, \quad (3.12)$$

where M denotes the mass matrix and is given by

$$M_{\mathbf{m}, \mathbf{m}'} = \int_{\Omega_e} l_{\mathbf{m}} l_{\mathbf{m}'} d\tilde{\mathbf{x}}. \quad (3.13)$$

The density gradient term is calculated as

$$\left(\frac{\partial \rho}{\partial \xi^j} \right)^e = d_j D_{\tilde{x}^j} \rho^e - \sum_{f'=1}^2 s_{\partial\Omega_{e,f'}} L_{\partial\Omega_{e,f'}} (\widehat{\rho} - \rho^e) \mathbf{n}_{\tilde{x}^j} \cdot \mathbf{n}. \quad (3.14)$$

Note that, in Eqs. (3.10), (3.11), and (3.14), $d_j = \partial \tilde{x}^j / \partial \xi^j$ and $s_{\partial\Omega_{e,f'}} = J_{\partial\Omega_{e,f'}} / J^E$ are constant values in the volume and surface integrals, respectively.

3.2 Temporal discretization

The semi-discretized equations in Eq. (3.4) can be represented as the following ordinary differential equation (ODE) system

$$\frac{d\mathbf{q}}{dt} = \mathcal{S}(\mathbf{q}, \nabla \mathbf{q}) + \mathcal{F}(\mathbf{q}, \nabla \mathbf{q}), \quad (3.15)$$

where $\mathcal{S}(\mathbf{q}, \nabla \mathbf{q})$ and $\mathcal{F}(\mathbf{q}, \nabla \mathbf{q})$ represent the tendencies with slow and fast contributions, respectively. This study adopted Runge–Kutta (RK) schemes to solve the ODE system from $t = n\Delta t$ to $t = (n+1)\Delta t$, where Δt is the time step and n is a natural number. In this subsection, we describe two approaches for temporal discretization, namely, horizontal explicit and vertical implicit (HEVI) and horizontal explicit and vertical explicit (HEVE) approaches.

HEVI approach

If the aspect ratio of horizontal grid spacing to its vertical counterpart is large, it is impractical to use fully explicit temporal schemes because the vertically propagating sound waves severely restrict the timestep. A strategy to avoid computational cost in such case is the HEVI approach. The terms corresponding to vertical dynamics with a fast time-scale are evaluated using an implicit temporal scheme, while the remaining terms are evaluated using an explicit temporal scheme. This procedure is regarded as a framework of implicit-explicit (IMEX) time integration scheme (Bao et al., 2015; Gardner et al., 2018). General formulation of IMEX RK scheme (e.g., Ascher et al., 1997) with ν stages can be represented as

$$\begin{aligned} \mathbf{q}^{(s)} &= \mathbf{q}^n + \Delta t \sum_{s'=1}^{s-1} a_{ss'} \mathcal{S}(t + c_{s'} \Delta t, \mathbf{q}^{(s')}) + \Delta t \sum_{s'=1}^s \tilde{a}_{ss'} \mathcal{F}(t + \tilde{c}_{s'} \Delta t, \mathbf{q}^{(s')}) \quad \text{for } s = 1, \dots, \nu \\ \mathbf{q}^{n+1} &= \mathbf{q}^n + \Delta t \sum_{s=1}^{\nu} b_s \mathcal{S}(t + c_s \Delta t, \mathbf{q}^{(s)}) + \Delta t \sum_{s=1}^{\nu} \tilde{b}_s \mathcal{F}(t + \tilde{c}_s \Delta t, \mathbf{q}^{(s)}), \end{aligned} \quad (3.16)$$

where $a_{ss'}$, b_s , and c_s define the explicit temporal integrator, while $\tilde{a}_{ss'}$, $\tilde{b}_s$, and $\tilde{c}_{s'}$ define the implicit temporal integrator; $c_s = \sum_{s'=1}^{s-1} a_{ss'}$ and $\tilde{c}_s = \sum_{s'=1}^{s-1} \tilde{a}_{ss'}$ represents time when slow and fast terms are evaluated, respectively. These coefficients are compactly represented using “double Butcher tableaux”, as shown in Table A.1. Note that, in the table of the explicit part, $\mathcal{A} = \{a_{ss'}\}$ with $a_{ss'} = 0$ for $s' \geq s$. On the other hand, for the implicit part, $\tilde{\mathcal{A}} = \{\tilde{a}_{ss'}\}$ with $\tilde{a}_{ss'} = 0$ for $s' > s$ in the case of the diagonally implicit RK scheme.

The terms associated with vertical mass flux, vertical pressure gradient, vertical flux of potential temperature, and buoyancy in Eq. (2.12) were treated as fast terms, whereas the other terms were treated as slow terms.

In SCALE-DG, several implicit and explicit (IMEX) RK schemes in Table 3.1 can be available. To minimize contaminating the spatial accuracy of high-order DGM by temporal errors present in low-order HEVI scheme, a third-order scheme proposed by Kennedy and Carpenter (2003) was adopted in our previous studies (Kawai and Tomita, 2025); it includes four explicit and three implicit evaluations. The corresponding double Butcher tableaux are given in Table A.1.

Table 3.1: Implicit and Explicit (IMEX) Runge–Kutta schemes available in HEVI temporal integration. For the number of RK stages, (I,E) represents implicit and explicit parts, respectively.

Abbrev.	Order	Num. of stages (I,E)	Note	Reference
IMEX_ARK232	2	(2,3)		Giraldo et al. (2013)
IMEX_ARK324	3	(3,4)		Kennedy and Carpenter (2003)

Table 3.2: Explicit Runge–Kutta schemes available in HEVE temporal integration

Abbrev.	Order	Num. of stages	Note	Reference
ERK_Euler	1	1	for debug	
ERK_SSP_2s2o	2	2	SSP	Shu and Osher (1988)
ERK_SSP_3s3o	3	3	SSP	Shu and Osher (1988)
ERK_SSP_4s3o	3	4	SSP	
ERK_SSP_5s3o_2N2*	3	5	SSP	Higueras and Roldán (2019)
ERK_RK4	4	4	classical RK4	
ERK_SSP_10s4o_2N	4	10	SSP	Ketcheson (2008)

In the implicit part of each stage, the corresponding nonlinear equation system is solved using Newton’s method. In each iteration, the linearized equation system is solved. Obtaining accurate solutions of the nonlinear equation system generally requires numerous iterations. However, this study performed a single iteration in Newton’s method (i.e., Rosenbrock approach), significantly reducing the computational cost. Similar approach has been used in previous studies (Ullrich and Jablonowski, 2012a). In the case of the collocation approach, because the horizontal dependency between all nodes within an element vanishes, the vertical implicit evaluation can be parallelly performed at each horizontal node.

For the case of HEVI, the volume and surface integrations in Eqs. (3.12) and (3.13) were evaluated using inexact integration with the LGL nodes. Consequently, $\mathbf{M}$ and $\mathbf{L}_{\partial\Omega_{e,3}}$ became diagonal matrices, which further simplified the matrix structure associated with the vertical spatial operator.

HEVE approach

When we consider a horizontal grid spacing with $O(10 \text{ m})$ such as in LES, the ratio of horizontal to vertical grid spacing approaches unity. The advantages of HEVI approach decrease. Thus, it is suitable to adopt a fully explicit temporal approach, referred to as HEVE approach. In such cases, RK schemes with a strong stability preserving (SSP) property (Gottlieb et al., 2001) are often used in combination with DGM. In SCALE-DG, several RK schemes in Table 3.2 can be available. The corresponding Butcher table and coefficients with Shu-Osher form are shown in Appendix A.1. In our previous studies (Kawai and Tomita, 2023, 2025), we adopted a ten-stage RK scheme with the fourth-order accuracy proposed by Ketcheson (2008).

When using the HEVE approach, entries of the matrices in Eqs. (3.12) and (3.13) were directly calculated following Sect. 3.2 in Hesthaven and Warburton (2007).

3.3 Discretization of tracer transport equations

A strategy to ensure non-negativity for tracer variables follows Light and Durran (2016). The element-mean non-negativity is satisfied by applying a flux-corrected transport (FCT) scheme at each stage in Runge–Kutta schemes. After the last stage, a nonlinear truncation and mass aware rescaling (TMAR) limiter is used to eliminate negatives at nodes within the element.

The discretization of tracer transport equations should be consistent to the mass conservation equation for the total density. Otherwise, consistency with continuity (CWC) would be violated, which may produce spurious sources and sinks of tracer mass. To satisfy of “narrow CWC”, mass flux temporally averaged during the dynamics is used.

3.4 Additional stabilization

3.4.1 Modal filtering

For high-order DGM, numerical instability is likely to occur in advection-dominated flows because the numerical dissipations with the upwind numerical fluxes weaken. Furthermore, we adopted a collocation approach due to its computational efficiency. One drawback is that the aliasing errors with evaluations of the nonlinear terms can drive numerical instability. To suppress this numerical instability, a modal filter was used as an additional stabilization mechanism. The filter matrix for the three-dimensional problem can be obtained as

$$\mathcal{F} = V^{3D} C^{3D} V^{3D}, \quad (3.17)$$

where V^{3D} represents the Vandermode matrix associated with the LGL interpolation nodes (in Eq. (3.3)) and C^{3D} represents the diagonal cutoff matrix. The entries of C^{3D} are defined as

$$C_{(m_1, m_2, m_3), (m'_1, m'_2, m'_3)}^{3D} = \delta_{m_1, m'_1} \sigma_{m_1}^h \delta_{m_2, m'_2} \sigma_{m_2}^h \delta_{m_3, m'_3} \sigma_{m_3}^v, \quad (3.18)$$

where σ_i^h and σ_i^v represent the decay coefficient for the one-dimensional horizontal and vertical modes i , respectively. Based on Hesthaven and Warburton (2007), a typical choice of the coefficient for mode i is provided with an exponential function as

$$\sigma_i = \begin{cases} 1 & \text{if } 0 \leq i \leq p_c \\ \exp \left[-\alpha_m \left(\frac{i - p_c}{p - p_c} \right)^{p_m} \right] & \text{if } p_c \leq i \leq p, \end{cases} \quad (3.19)$$

where p_c , p_m , and α_m represent the cutoff parameter, the order of the filter, and the non-dimensional decay strength, respectively. In this study, p_c was considered 0. We applied the filter $\mathcal{F}$ to the solution vector $\mathbf{q}$ (in Eq. (2.13)) at the final stage of the RK scheme with a timestep Δt . Then, the decay time scale for the highest mode can be regarded as approximately equal to $\Delta t / \alpha_m$. We set the order p_m , and decay coefficient α_m such that the strength of filter should ensure numerical stability while being as weak as possible.

3.4.2 Buoyancy term

The balance between the pressure gradient and buoyancy terms should be carefully treated in the discrete momentum equation (e.g., Blaise et al., 2016; Orgis et al., 2017). In the

above formulation, because a different discretization space is used between the terms, a numerical imbalance is possible and may cause spurious oscillations, which can destabilize the simulations. To avoid this incompatibility, the vertical polynomial order for the density in the buoyancy term was reduced by one following Blaise et al. (2016).

4 Physical parameterization

4.1 Turbulence

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This section describes the subgrid-scale (SGS) turbulence model used for large-eddy simulation (LES) in SCALE-DG. In LES, the large energy-containing turbulent eddies are explicitly resolved by the numerical model, while the effects of unresolved SGS motions are represented through eddy-viscosity models.

4.1.1 Smagorinsky-type model

As a turbulent model, this subsection describes a Smagorinsky–Lilly type model (Smagorinsky, 1963; Lilly, 1962) that considered the stratification effect (Brown et al., 1994). As a spatial filter, the Favre-filtering (Favre, 1983) was used. We do not explicitly denote the symbol representing the spatial filter because the filtering approach is essentially the same as that explained in Appendix A of Kawai and Tomita (2023). The difficulties in the derivation of viscous and diffusion terms are caused by the gradient of vector quantities and the spatial divergence with the non-orthogonal basis because the manipulations grow increasingly complex. However, previous studies that utilized tensor analysis help us provide a systematic derivation (e.g., Ullrich, 2014; Rančić et al., 2017). In the absence of a vertical coordinate transformation, the parameterized fluxes with the turbulent model can be represented in the general curvilinear coordinates as

$$\mathbf{f}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\rho\tau^{11} \\ -\sqrt{G}\rho\tau^{12} \\ -\sqrt{G}\rho\tau^{13} \\ -\sqrt{G}\rho\tau_*^1 \end{pmatrix}, \quad \mathbf{g}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\rho\tau^{21} \\ -\sqrt{G}\rho\tau^{22} \\ -\sqrt{G}\rho\tau^{23} \\ -\sqrt{G}\rho\tau_*^2 \end{pmatrix}, \quad (4.1)$$

$$\mathbf{h}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\rho\tau^{31} \\ -\sqrt{G}\rho\tau^{32} \\ -\sqrt{G}\rho\tau^{33} \\ -\sqrt{G}\rho\tau_*^3 \end{pmatrix}, \quad (4.2)$$

and the source term can be given by

$$\mathbf{S}_{\text{SGS}}(\mathbf{q}, \nabla \mathbf{q}) = \begin{pmatrix} 0 \\ -\sqrt{G}\Gamma_{ml}^1 \rho \tau^{ml} \\ -\sqrt{G}\Gamma_{ml}^2 \rho \tau^{ml} \\ -\sqrt{G}\Gamma_{ml}^3 \rho \tau^{ml} \\ 0 \end{pmatrix}. \quad (4.3)$$

In the equations, τ^{ij} is the contravariant components of parameterized eddy viscous flux tensor ($i = 1, 2, 3$ and $j = 1, 2, 3$) and can be written as

$$\tau^{ij} = -2\nu_{\text{SGS}} \left(S^{ij} - \frac{G^{ij}}{3} D \right) - \frac{2}{3} G^{ij} K_{\text{SGS}}, \quad (4.4)$$

where S^{ij} is the strain velocity tensor, ν_{SGS} is the eddy viscosity, D is the divergence of the three-dimensional velocity, and K_{SGS} is the SGS kinetic energy. The strain velocity tensor is represented as

$$S^{ij} = \frac{1}{2} \left(G^{im} \frac{\partial u_{,m}^j}{\partial \xi^m} + G^{jm} \frac{\partial u_{,m}^i}{\partial \xi^m} \right), \quad (4.5)$$

using the covariant derivative of the contravariant velocity component

$$u_{,j}^i = \frac{\partial u^i}{\partial \xi^j} + u^m \Gamma_{jm}^i. \quad (4.6)$$

The eddy viscosity is written as

$$\nu_{\text{SGS}} = C_s \Delta_{\text{SGS}} |S|, \quad (4.7)$$

where C_s , Δ_{SGS} , and $|S|$ represent the Smagorinsky constant, the filter length, and the norm of strain tensor defined as $\sqrt{2G_{im}G_{jn}S^{ij}S^{mn}}$, respectively. The parameterized eddy diffusive flux can be written as

$$\tau_*^i = -\nu_{\text{SGS}}^* G^{ij} \frac{\partial \theta}{\partial \xi^j}, \quad (4.8)$$

where ν_{SGS}^* is the eddy diffusion coefficient. For further details of the turbulent model, refer to Sect. 2.2 of Nishizawa et al. (2015).

4.1.2 Common formulation

Although the formulations of the SGS turbulence models may differ, they all provide the parameterized SGS momentum and scalar fluxes required by the governing equations. Once evaluated, these fluxes are incorporated into the governing equations as viscous and diffusive flux terms. Their spatial discretization follows the common DG formulation for the diffusion terms described in Sect. 3.1.

The SGS flux terms are treated explicitly in time. In the present implementation, their tendencies are integrated using the forward Euler method. Therefore, the timestep must satisfy the stability constraint associated with the SGS eddy viscosity and diffusivity.

4.2 Boundary layer turbulence model

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4.2.1 Basic formulation of PBL parameterizations

Unlike the SGS turbulence models described in the previous section, PBL parameterizations represent the turbulent transport associated with the entire planetary boundary layer (PBL), including energetic turbulent eddies that are not explicitly resolved by mesoscale atmospheric simulations. Under the assumption of local horizontal homogeneity, the resolved mean fields are considered to vary more weakly in the horizontal directions than in the vertical direction over the characteristic scales of boundary-layer turbulence. Conventional PBL parameterizations are therefore commonly formulated as one-dimensional vertical-column models. The principal effects of unresolved turbulence are represented through vertical fluxes of momentum, heat, moisture, and other scalars. Horizontal subgrid-scale mixing, when required, is generally treated separately.

To describe the evolution of the resolved mean state, the governing equations are Reynolds-averaged using the Reynolds decomposition as $\phi = \bar{\phi} + \phi'$ where the overbar and prime denote the ensemble-mean and turbulent components, respectively. The mean horizontal momentum equations contain the vertical turbulent momentum fluxes as

$$\frac{\partial \bar{u}}{\partial t} = -\frac{\partial \overline{u'w'}}{\partial z} + \mathcal{F}_u, \quad (4.9)$$

$$\frac{\partial \bar{v}}{\partial t} = -\frac{\partial \overline{v'w'}}{\partial z} + \mathcal{F}_v, \quad (4.10)$$

where $\mathcal{F}_u$ and $\mathcal{F}_v$ denote the resolved dynamical forcing terms. Similarly, the mean equation for a scalar quantity ϕ is written as

$$\frac{\partial \bar{\phi}}{\partial t} = -\frac{\partial \overline{\phi'w'}}{\partial z} + \mathcal{F}_\phi. \quad (4.11)$$

These equations are not closed because the turbulent fluxes, such as $\overline{u'w'}$ and $\overline{\phi'w'}$, are additional unknowns, which constitutes the central problem of turbulence closure.

In first-order closure models, the turbulent fluxes are directly assumed to be proportional to the corresponding gradients of the resolved mean fields. In higher-order closure models, including the Mellor–Yamada (MY) family, the turbulent fluxes are instead determined from prognostic or diagnostic equations for second-order turbulent moments. These equations describe the balances among shear production, buoyancy production or destruction, pressure correlations, turbulent transport, and viscous dissipation. After suitable closure approximations are introduced, the second-moment equations form a closed system.

In the MY level-2 closure, the local time tendencies and turbulent transport terms in the second-moment equations are neglected. The second moments are therefore assumed to be in local equilibrium, and their governing equations reduce to a set of algebraic relations. By solving these relations, the vertical turbulent fluxes can be represented by the local gradients of the resolved mean fields as

$$\overline{u'w'} = -K_M \frac{\partial \bar{u}}{\partial z}, \quad (4.12)$$

$$\overline{\phi'w'} = -K_H \frac{\partial \bar{\phi}}{\partial z}, \quad (4.13)$$

where K_M and K_H denote the eddy viscosity and eddy diffusivity, respectively. Thus, in the MY framework, the gradient-diffusion form is not merely imposed as an independent empirical assumption, but arises as a constitutive relation obtained from the locally equilibrated second-moment equations. The eddy viscosity and diffusivity are written as

$$K_M = lqS_M, \quad (4.14)$$

$$K_H = lqS_H, \quad (4.15)$$

where l is the turbulent mixing length, $q = \sqrt{\overline{u'^2} + \overline{v'^2} + \overline{w'^2}} = \sqrt{2e}$ is the turbulent velocity scale, and S_M and S_H are stability functions for momentum and scalar transport, respectively. The factors lq provide the characteristic turbulent diffusivity scale, whereas the stability functions arise from the algebraic solution of the second-moment equations and represent the effects of shear and buoyancy on the turbulent fluxes. Under unstable stratification, buoyancy production enhances turbulent motions and generally increases the mixing coefficients. Under stable stratification, buoyancy suppresses vertical motions and reduces turbulent transport. Different PBL parameterization schemes therefore differ mainly in the closure assumptions used to determine l , q , S_M , and S_H , and in whether these quantities are diagnosed or predicted.

4.2.2 Mellor–Yamada Nakanishi–Niino (MYNN) level 2.0 model

This model employs the local-equilibrium form of the MYNN second-moment closure (Nakanishi and Niino, 2006). In this model, the turbulent velocity scale q is diagnosed from the local balance of turbulent kinetic energy (TKE), rather than predicted from a time-dependent TKE equation. The local TKE balance is written as

$$P + B - \varepsilon = 0, \quad (4.16)$$

where P is the shear production, B is the buoyancy production, and ε is the dissipation rate of TKE. For a horizontally homogeneous flow, these terms are expressed as

$$P = \frac{\partial \bar{u}}{\partial z} \overline{u'w'} + \frac{\partial \bar{v}}{\partial z} \overline{v'w'}, \quad B = -\frac{g}{\theta_0} \overline{\theta'w'}, \quad \varepsilon = \frac{q^3}{B_1 l}, \quad (4.17)$$

where θ_0 is a reference potential temperature and B_1 is a closure constant. Using the gradient-diffusion relations in Eqs. (4.12) and (4.13), the production terms become

$$P \sim K_M S^2, \quad B \sim -K_H N^2, \quad (4.18)$$

where S^2 and N^2 are the squared magnitude of the vertical wind shear and the squared Brunt–Väisälä frequency, respectively.

Substituting $K_M = lqS_M$ and $K_H = lqS_H$ into the local TKE balance provides

$$lq (S_M S^2 - S_H N^2) - \frac{q^3}{B_1 l} = 0. \quad (4.19)$$

The diagnosed turbulent velocity scale therefore satisfies

$$q^2 = B_1 l^2 S_M S^2 (1 - R_f) \quad (4.20)$$

where R_f is the flux Richardson number as

$$R_f = \frac{S_H N^2}{S_M S^2} = \frac{K_H N^2}{K_M S^2}. \quad (4.21)$$

The stability functions are obtained by solving the locally equilibrated algebraic equations for the Reynolds stresses, turbulent heat fluxes, and scalar variance. Although the detailed elimination of the individual second moments is omitted here, the resulting functions can be expressed in the form

$$S_H = S_H(R_f), \quad S_M = S_M(R_f). \quad (4.22)$$

Thus, the effects of shear and stratification on momentum and heat transport are represented through a single nondimensional parameter, R_f .

The flux Richardson number is related to the gradient Richardson number $Ri = N^2/S^2$ through the stability functions as

$$Ri = R_f \frac{S_M(R_f)}{S_H(R_f)}. \quad (4.23)$$

Because the ratio S_M/S_H is itself a known function of R_f , this algebraic relation can be inverted to obtain

$$R_f = R_f(Ri), \quad (4.24)$$

which corresponds to solving a quadratic equation for R_f .

The diagnostic procedure of the MYNN level 2.0 closure may therefore be summarized as

$$Ri; \longrightarrow; R_f; \longrightarrow; (S_M, S_H); \longrightarrow; q; \longrightarrow; (K_M, K_H). \quad (4.25)$$

Ri is first calculated from the resolved vertical gradients. The flux Richardson number is then obtained from the algebraic MYNN relation in Eq. (4.24), followed by the evaluation of the stability functions in Eq. (4.22). The turbulent velocity scale is diagnosed from Eq. (4.20), and the eddy viscosity and diffusivity are finally evaluated using Eq. (4.15). The mixing length l is specified through a separate MYNN mixing-length formulation.

The complete algebraic formulation, including the closure constants and analytical expressions for the stability functions, is given in Appendix A of Nakanishi and Niino (2006).

4.2.3 Common formulation

Although different PBL parameterization schemes employ different formulations for the turbulent mixing coefficients, their effects are incorporated into the governing equations in SCALE-DG through a common vertical diffusion operator. For momentum, potential temperature, moisture, and other prognostic scalars, the turbulent fluxes are represented by the gradient-diffusion relations, and the corresponding tendency is written in the generic form

$$\frac{\partial \rho \phi}{\partial t} = \frac{\partial}{\partial z} \left(\rho K_\phi \frac{\partial \phi}{\partial z} \right), \quad (4.26)$$

where K_ϕ denotes the turbulent diffusion coefficient determined by the selected PBL parameterization.

The vertical diffusion operators are spatially discretized using the symmetric interior penalty (SIP) formulation (Arnold et al., 2002). This formulation weakly couples neighboring DG elements through consistent numerical fluxes and penalty terms, providing a stable and high-order discretization of the diffusion operator. We evaluate the turbulent diffusion coefficients at the LGL nodes and enter the SIP discretization as spatially varying diffusion coefficients.

The characteristic time scale associated with vertical turbulent diffusion is often much shorter than that of the resolved atmospheric dynamics, especially under strongly mixed boundary-layer conditions. To avoid the severe timestep restriction imposed by explicit integration, the vertical diffusion terms are integrated implicitly using the backward Euler method.

4.3 Microphysics

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To represent cloud-microphysics processes, we provide two microphysics schemes using SCALE library: three-class one-moment bulk and six-class one-moment bulk schemes.

In addition, a large-scale condensation scheme can be used. It is useful to investigate a feedback mechanism between the equations of motion and moisture within simplified physics.

4.3.1 Three-class one-moment bulk scheme

As a three-class one-moment bulk microphysics scheme, the Kessler parameterization (Kessler, 1969) can be used. In this scheme, the distribution function of particle size is expressed only by mass concentration. Considering two categories of water in cloud and rain, the ratios of the densities of cloud and rain to total air density are calculated.

The details of formulation are found in Section 8.3.1 of SCALE description (Nishizawa et al., 2025).

4.3.2 Six-class one-moment bulk scheme

A six-class one-moment bulk microphysics scheme is based on Tomita (2008). This scheme calculates mass exchange among six categories of water substances, which are water vapor, cloud water, rain, cloud ice, snow, and graupel. Although the scheme largely follows Lin et al. (1983), the original method of Lin et al. (1983) is modified as follows: Both cloud water and cloud ice are generated only by a saturation adjustment process, and wet growth process of graupel is omitted. Tomita (2008) indicates that these modifications result in 20% reduction in computational cost without significant changes on physical performance.

The details of formulation are found in Section 8.3.2 of SCALE description (Nishizawa et al., 2025).

4.3.3 Large-scale condensation scheme

If the moisture content exceeds its saturation value, the excess water vapor is converted to cloud water. The associated latent heat is released to the atmosphere and the cloud water is removed with a certain timescale. A typical value of timescale is set to about $O(10^3)$ seconds when the horizontal grid spacing is $O(10^2)$ km.

Based on approaches in some AGCMs, such as the Integrated Forecasting System (IFS) of the European Centre for Medium-Range Weather Forecasts (Tiedtke, 1987), the formulation is described as follows. If the specific humidity exceeds the saturation value $q_{v,s}$, specific humidity q_v is reduced toward $q_{v,s}$. The excess water vapor is converted to cloud water, which is removed as precipitation. The temperature is increased by the release of latent heat.

During the moist adjustment, the moist internal energy is assumed to be conserved:

$$\delta(C_{v,\text{tot}}T + Lq_v + L_lq_c) = 0, \quad (4.27)$$

where $C_{v,\text{tot}} = (1 - q_v - q_c)C_{v,d} + q_vC_{v,v} + q_cC_{v,l}$, and L and L_l denote the latent heats of vaporization and fusion, respectively. Following the definition adopted in the SCALE

library, the moist internal energy is defined by taking liquid water as the reference state and, thus, $L_l = 0$. Since cloud water is produced by condensation of water vapor, $\delta q_c = -\delta q_v$. Retaining terms to first order in the adjustment increments, we obtain

$$C_{v,\text{tot}}^* \delta T = [L + (C_{v,v} - C_{v,l})T^*] \delta q_c, \quad (4.28)$$

where the single asterisks denote the unadjusted states. The amount of condensed water is determined by the difference between the unadjusted specific humidity and the saturation specific humidity at the unadjusted temperature:

$$\delta q_c = q_v^* - q_{v,s}(T^* + \delta T). \quad (4.29)$$

Because the $q_{v,s}$ depends on temperature through the Clausius-Clapeyron equation, Eqs. (4.28) and (4.29) form an implicit system. For idealized experiments, $q_{v,s}$ is often linearized around T^* as

$$q_{v,s}(T^* + \delta T) \approx q_{v,s}(T^*) + \left. \frac{dq_{v,s}}{dT} \right|_{T=T^*} \delta T. \quad (4.30)$$

Neglecting the pressure dependence during the adjustment, the temperature derivative of the saturation specific humidity is approximated using the Clausius-Clapeyron relation as

$$\frac{dq_{v,s}}{dT} \simeq \frac{Lq_{v,s}}{R_v T^2}.$$

Combining Eqs. (4.28) and (4.30) gives

$$\delta q_c = C_{v,\text{tot}}^* \frac{q_v^* - q_{v,s}(T^*)}{C_{v,\text{tot}}^* + [L + (C_{v,v} - C_{v,l})T^*] \left. \frac{dq_{v,s}^*}{dT} \right|_{T=T^*}}. \quad (4.31)$$

When a simplified moist thermodynamics is used, $C_{v,v} = C_{v,l}$ and $C_{v,\text{tot}}^* \approx C_{v,d}$. Then, δq_c simplifies to

$$\delta q_c = \frac{q_v^* - q_{v,s}(T^*)}{1 + \left. \frac{L}{C_{v,d}} \frac{dq_{v,s}^*}{dT} \right|_{T=T^*}}, \quad (4.32)$$

Here, δq_c is introduced as an intermediate quantity representing the amount of condensed water. The condensate is subsequently removed as precipitation and is not retained as prognostic cloud water in this scheme.

Finally, the condensation rate is defined as

$$C_{\text{rate}} = \frac{\delta q_c}{\tau_{\text{LSC}}}$$

where τ_{LSC} is the prescribed adjustment timescale for the large-scale condensation process. For the diabatic heating in the density-weighted potential temperature in Eq. (2.25) and the source terms in the mass equations of water vapor, we have

$$Q_{\text{LSC}} = \rho L C_{\text{rate}}, \quad (4.33)$$

$$S_{\rho q_v, \text{LSC}} = -\rho C_{\text{rate}}. \quad (4.34)$$

4.4 Cumulus parameterization

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Cumulus parameterization represents the effects of unresolved moist convection on the resolved atmospheric state. In SCALE-DG, a moist convective adjustment scheme is available as a simple representation of cumulus convection.

4.4.1 Moist convective adjustment

The scheme is based on the concept of Manabe et al. (1965). Saturated convective instability is diagnosed from the vertical distribution of saturated moist static energy (MSE) in nearly saturated layers. If such instability is detected, the unstable layer is adjusted toward a moist-neutral thermodynamic profile subject to column-integrated MSE conservation and the available-water constraint. Water vapor condensed during the adjustment is immediately removed as precipitation. The adjustment is repeated until no saturated convective instability remains in the column. The overall procedure of the adjustment is summarized in Fig. 4.1.

The saturated MSE is defined as

$$h_s = C_p(q_{v,s})T + gz + Lq_{v,s}, \quad (4.35)$$

where $C_p(q_v) = (1 - q_v)C_{p,d} + q_vC_{p,v}$ and $q_{v,s}$ is the saturation specific humidity. A layer is regarded as potentially convectively unstable when it is nearly saturated and h_s decreases with height, i.e.,

$$\frac{\partial h_s}{\partial z} < 0. \quad (4.36)$$

In practice, small thresholds are introduced for relative humidity and the vertical difference of h_s in order to avoid triggering the adjustment by numerical noise. The contiguous layer satisfying these criteria is referred to as the *diagnosed unstable core* $\mathcal{C}$. The *adjustment region* $\mathcal{A}$ is the layer over which the moist-neutral profile and the column-integrated constraints are imposed. It initially corresponds to $\mathcal{C}$ but may be extended vertically when necessary to obtain a water-feasible and energy-conserving adjusted state.

For a given $\mathcal{A}$, the adjusted thermodynamic profile is determined using the temperature T_b at the base of $\mathcal{A}$ as a single scalar degree of freedom. For each trial value of T_b , a moist-neutral temperature profile is constructed by requiring vertically uniform saturated MSE:

$$h_s(\tilde{T}(z; T_b), \tilde{p}(z; T_b), z) = h_{s,0}, \quad (4.37)$$

where a tilde denotes a trial state before the diagnosed precipitation is removed. Equation (4.37) is integrated upward from T_b to obtain $\tilde{T}(z; T_b)$.

The corresponding water-vapor profile is constructed subject to the saturation constraint. Nodes that are nearly saturated within $\mathcal{C}$ are adjusted to saturation,

$$\tilde{q}_v(z; T_b) = q_{v,s} \left(\tilde{T}(z; T_b), \tilde{p}(z; T_b) \right), \quad (4.38)$$

whereas at the other nodes within $\mathcal{A}$,

$$\tilde{q}_v(z; T_b) = \min \left[q_v^*, q_{v,s} \left(\tilde{T}(z; T_b), \tilde{p}(z; T_b) \right) \right]. \quad (4.39)$$

Here, the asterisk denotes the state before the adjustment. During this trial-profile construction, the density ρ^* is held fixed, while the pressure is updated through

$$\tilde{p} = \rho^* R(\tilde{q}_v) \tilde{T}, \quad R(q_v) = (1 - q_v) R_d + q_v R_v.$$

The moist-neutral profile and pressure are determined consistently by iteration.

For each T_b , the column-integrated water vapor and MSE of the resulting trial profile are

$$W(T_b) = \int_{\mathcal{A}} \rho^* \tilde{q}_v(z; T_b) dz, \quad H(T_b) = \int_{\mathcal{A}} \rho^* [C_p(\tilde{q}_v) \tilde{T} + gz + L \tilde{q}_v] dz,$$

and their values before the adjustment are

$$W^* = \int_{\mathcal{A}} \rho^* q_v^* dz, \quad H^* = \int_{\mathcal{A}} \rho^* [C_p(q_v^*) T^* + gz + L q_v^*] dz.$$

The available-water constraint first restricts the admissible range of T_b according to

$$W(T_b) \leq W^*. \quad (4.40)$$

Within this range, T_b is determined by imposing conservation of the column-integrated MSE:

$$H(T_b) = H^*. \quad (4.41)$$

Thus, the water constraint restricts the admissible family of trial profiles, and the MSE constraint selects the final value of T_b . If no such solution exists within the initially diagnosed unstable core, $\mathcal{A}$ is progressively extended vertically and the calculation is repeated. This extension is a numerical procedure for obtaining a consistent adjusted state and does not represent physical entrainment.

Once T_b has been determined, the reduction in column-integrated water vapor is immediately removed as precipitation:

$$P_{\text{MCA}} = W^* - W(T_b) = \int_{\mathcal{A}} \rho^* (q_v^* - \tilde{q}_v) dz \geq 0. \quad (4.42)$$

The precipitation mass is distributed within $\mathcal{A}$ over nodes where the local water-vapor mass density decreases. No condensate is retained as a prognostic cloud-water variable.

After the state has been updated, saturated convective instability is diagnosed again over the entire column. If another unstable core is found, the adjustment procedure described above is repeated using the updated state. This diagnosis–adjustment cycle continues until no saturated convective instability remains in the column. The precipitation produced in successive adjustments is accumulated throughout this cycle.

Let $\rho_{\text{precip}}(z)$ denote the accumulated local precipitation mass density after completion of the diagnosis–adjustment cycle. The final density is obtained as

$$\rho^{\text{adj}} = \rho^* - \rho_{\text{precip}}.$$

The specific humidity changes consistently with the reduced density, while the water-vapor mass density remains unchanged during this final precipitation-removal step. The pressure, potential temperature, and other thermodynamic quantities are then recomputed consistently to obtain the final adjusted state.

The adjustment is applied over one physics time step Δt_{MCA} . The tendencies are

$$S_{\rho, \text{MCA}} = \frac{\rho^{\text{adj}} - \rho^*}{\Delta t_{\text{MCA}}}, \quad (4.43)$$

$$S_{\rho\theta, \text{MCA}} = \frac{\rho^{\text{adj}}\theta^{\text{adj}} - \rho^*\theta^*}{\Delta t_{\text{MCA}}}, \quad (4.44)$$

$$S_{\rho q_v, \text{MCA}} = \frac{\rho^{\text{adj}}q_v^{\text{adj}} - \rho^*q_v^*}{\Delta t_{\text{MCA}}}. \quad (4.45)$$

The precipitation mass flux and the associated energy flux removed from the atmospheric column are diagnosed from the precipitation due to the adjustment.

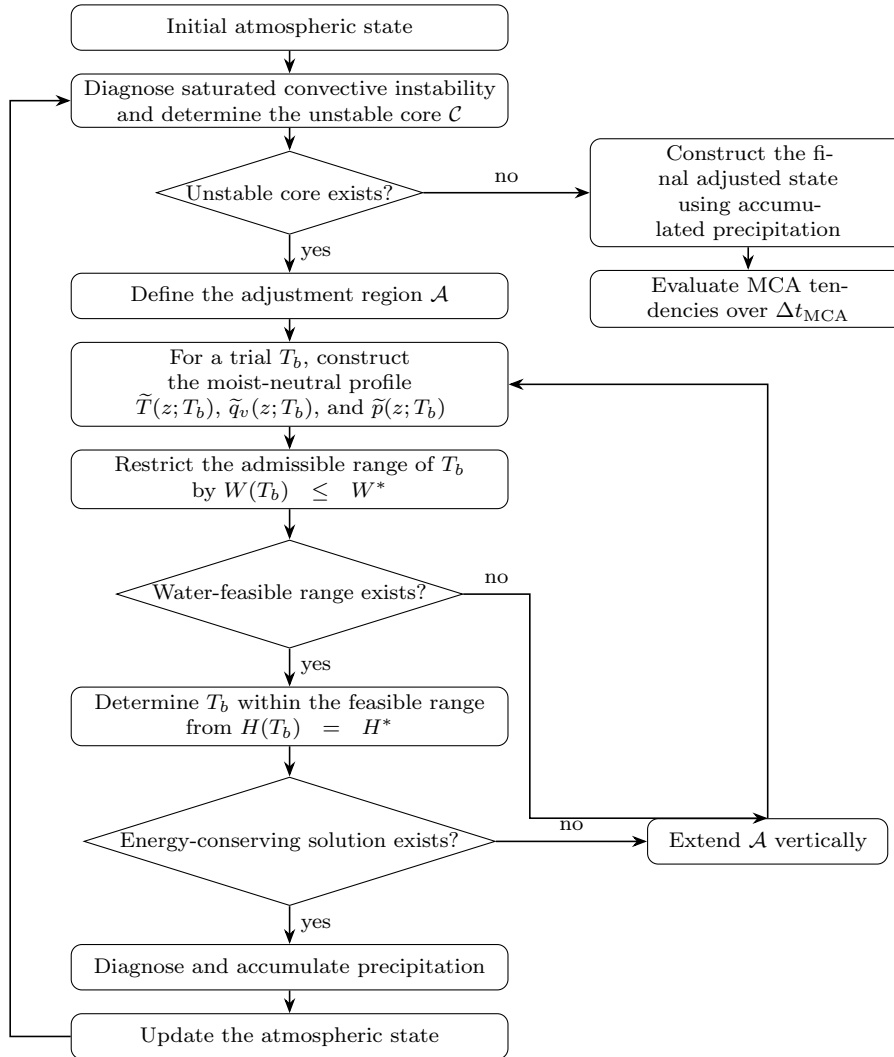


Figure 4.1: Schematic procedure of the moist convective adjustment scheme. The available-water constraint first restricts the admissible range of T_b , and the MSE constraint then determines T_b within this range. The diagnosis–adjustment cycle is repeated until no saturated convective instability remains in the column.

4.5 Radiation

Corresponding author : Yuta Kawai

Radiative transfer describes the propagation of electromagnetic radiation through the atmosphere and determines the radiative heating and cooling resulting from the absorption, emission, and, in general, scattering of radiation.

4.5.1 Basic formulation of radiative transfer

The following formulation follows the standard theory of radiative transfer (e.g., Liou, 2002). In general, the specific intensity depends on the position, propagation direction, wavelength, and time. For a given wavelength and propagation direction, the radiative transfer equation is written as

$$\frac{dI_\lambda(\boldsymbol{\sigma})}{ds} = -\chi_\lambda I_\lambda(\boldsymbol{\sigma}) + \kappa_\lambda B_\lambda + \sigma_{s,\lambda} \mathcal{S}_\lambda(\boldsymbol{\sigma}), \quad (4.46)$$

where $I_\lambda(\boldsymbol{\sigma})$ is the specific intensity at wavelength λ in the propagation direction $\boldsymbol{\sigma}$, s is the distance along the radiation path, κ_λ and $\sigma_{s,\lambda}$ are the absorption and scattering coefficients, respectively, $\chi_\lambda = \kappa_\lambda + \sigma_{s,\lambda}$ is the extinction coefficient, and B_λ is the Planck function. $\mathcal{S}_\lambda$ is the scattering source function defined as

$$\mathcal{S}_\lambda(\boldsymbol{\Omega}) = \frac{1}{4\pi} \int_{4\pi} I_\lambda(\boldsymbol{\Omega}') P_\lambda(\boldsymbol{\Omega}, \boldsymbol{\Omega}') d\boldsymbol{\Omega}', \quad (4.47)$$

where $P_\lambda(\boldsymbol{\Omega}, \boldsymbol{\Omega}')$ is the scattering phase function that describes the angular distribution of scattered radiation. Introducing the optical thickness τ_λ defined as

$$d\tau_\lambda = \chi_\lambda ds, \quad (4.48)$$

Eq. (4.46) becomes

$$\frac{dI_\lambda}{d\tau_\lambda} = -I_\lambda + S_\lambda. \quad (4.49)$$

Here, S_λ is the source function defined as

$$S_\lambda = \frac{\kappa_\lambda B_\lambda + \sigma_{s,\lambda} J_\lambda}{\chi_\lambda} \quad (4.50)$$

$$= (1 - \omega_\lambda) B_\lambda + \omega_\lambda J_\lambda, \quad (4.51)$$

where $\omega_\lambda = \sigma_{s,\lambda}/\chi_\lambda$ is the single scattering albedo.

In atmospheric simulations, the radiative transfer is typically solved independently in each atmospheric column under the plane-parallel approximation, whereby horizontal radiative transport is neglected. We introduce the optical thickness along the vertical direction z as

$$d\tau_{\lambda,z} = -\chi_\lambda dz = -\chi_\lambda \mu ds = -\mu d\tau_\lambda, \quad (4.52)$$

where μ is the cosine of the zenith angle θ . Thus, we can rewrite Eq. (4.49) as

$$\mu \frac{dI_\lambda}{d\tau_{\lambda,z}} = I_\lambda - S_\lambda. \quad (4.53)$$

The equation for the radiative fluxes is obtained by integrating Eq. (4.53) over the semi-sphere as

$$\frac{dF_{\lambda,z}^{\uparrow}}{d\tau_{\lambda,z}} = 4\pi J_{\lambda}^{\uparrow} - 2\pi(1 - \omega_{\lambda})B_{\lambda} - \frac{\omega_{\lambda}}{2} \int_0^1 d\mu \int_{4\pi} P_{\lambda}(\mathbf{\Omega}, \mathbf{\Omega}') I_{\lambda}(\mathbf{\Omega}') d\Omega', \quad (4.54)$$

$$\frac{dF_{\lambda,z}^{\downarrow}}{d\tau_{\lambda,z}} = -4\pi J_{\lambda}^{\downarrow} + 2\pi(1 - \omega_{\lambda})B_{\lambda} + \frac{\omega_{\lambda}}{2} \int_{-1}^0 d\mu \int_{4\pi} P_{\lambda}(\mathbf{\Omega}, \mathbf{\Omega}') I_{\lambda}(\mathbf{\Omega}') d\Omega'. \quad (4.55)$$

Here, superscripts $\uparrow$ and $\downarrow$ indicate the quantities integrated over the upward and downward hemispheres, respectively, J_{λ} is the mean intensity defined as

$$J_{\lambda} = \frac{1}{4\pi} \int_{4\pi} I_{\lambda}(\mathbf{\Omega}) d\Omega, \quad (4.56)$$

and the upward and downward fluxes are defined as

$$F_{\lambda}^{\uparrow} = 2\pi \int_0^1 I_{\lambda}(\mu) \mu d\mu, \quad F_{\lambda}^{\downarrow} = 2\pi \int_{-1}^0 I_{\lambda}(\mu) |\mu| d\mu, \quad (4.57)$$

where both fluxes are defined as positive quantities.

The hemispherically integrated equations are not closed because they still contain the angular moments of the radiation field and the scattering integral. In the two-stream approximation (Toon et al., 1989), the angular dependence within each hemisphere is represented by a single effective propagation direction. The resulting angular integration is approximated using a diffusivity factor D_{λ} , which accounts for the increased optical path length of obliquely propagating radiation. Then, the two-stream equations can therefore be written schematically as

$$\frac{dF_{\lambda,z}^{\uparrow}}{d\tau_{\lambda,z}} = D_{\lambda}^{\uparrow} (F_{\lambda}^{\uparrow} - \pi S_{\lambda}^{\uparrow}), \quad (4.58)$$

$$\frac{dF_{\lambda,z}^{\downarrow}}{d\tau_{\lambda,z}} = -D_{\lambda}^{\downarrow} (F_{\lambda}^{\downarrow} - \pi S_{\lambda}^{\downarrow}), \quad (4.59)$$

where $D_{\lambda}^{\uparrow}$ and $D_{\lambda}^{\downarrow}$ are the diffusivity factors determined by the adopted angular quadrature, $S_{\lambda}^{\uparrow}$ and $S_{\lambda}^{\downarrow}$ denote the corresponding hemispherically averaged source functions. Note that the source term πS_{λ} is not obtained by directly evaluating $2\pi \int_0^1 S_{\lambda} d\mu$. Rather, it appears as part of the angular closure introduced in the two-stream approximation.

In the current version of SCALE-DG, the spectral dependence is treated separately in each radiation scheme by dividing the spectrum into one or more bands. Furthermore, the scattering is neglected in all radiation schemes. Consequently, $\sigma_{s,\lambda} = 0$ and $\chi_{\lambda} = \kappa_{\lambda}$, and the source function is simplified as $S_{\lambda} = B_{\lambda}$. The radiation transfer equation in Eq. (4.59) using the two-stream approximation therefore becomes

$$\frac{dF_{\lambda,z}^{\uparrow}}{d\tau_{\lambda,z}} = D_{\lambda}^{\uparrow} (F_{\lambda}^{\uparrow} - \pi B_{\lambda}), \quad (4.60)$$

$$\frac{dF_{\lambda,z}^{\downarrow}}{d\tau_{\lambda,z}} = -D_{\lambda}^{\downarrow} (F_{\lambda}^{\downarrow} - \pi B_{\lambda}). \quad (4.61)$$

The following sections describe the radiation schemes currently available in SCALE-DG. Although the parameterizations of optical properties differ among the schemes, all are based on Eq. (4.61).

4.5.2 Gray radiation scheme

As the simplest radiation model incorporating moisture feedback, a gray radiation model is provided. This scheme is gray in infrared so that a single optical thickness is defined for the entire longwave spectrum, which includes a parameterization of long-wave absorption by CO_2 .

Following Eq. (5) in Vallis et al. (2018), the vertical gradient of longwave optical thickness is parameterized as

$$\frac{\partial \tau_{\text{LW}}}{\partial \sigma} = a_{\text{LW}} \mu + b_{\text{LW}} q_v + c_{\text{LW}} \ln \left(\frac{C_{\text{CO}_2}}{360} \right), \quad (4.62)$$

where $\sigma = p/P_0$, q_v is the specific humidity, C_{CO_2} is the atmospheric CO_2 concentration [ppm], μ is a prescribed scaling factor, and a_{LW} , b_{LW} , and c_{LW} are empirical coefficients.

Byrne and O’Gorman (2013) used $(a_{\text{LW}}, b_{\text{LW}}, c_{\text{LW}}) = (0.8678, 1997.9, 0)$ with the coefficients obtained by fitting this parameterization to the longwave optical depths employed by Frierson et al. (2006). Their idealized experiments used a planetary albedo of approximately 0.38. For experiments employing an albedo closer to that of Earth, approximately 0.30, Vallis et al. (2018) suggested $(a_{\text{LW}}, b_{\text{LW}}, c_{\text{LW}}) = (0.1627, 1997.9, 0.17)$. The latter parameter set also includes the explicit parameterization of longwave absorption by CO_2 , which was derived using output from the Santa Barbara DISORT Atmospheric Radiative Transfer model (SBDART) (Ricchiazzi et al., 1998).

SCALE-DG uses the coefficient set of Vallis et al. (2018) by default, while the Byrne and O’Gorman (2013) set is also available as an alternative.

4.5.3 Two-band infrared and one-band solar radiation scheme

This scheme has two infrared bands and one shortwave band, as described in Geen et al. (2016), which provides an intermediate complexity between gray radiation and more sophisticated radiation schemes. All bands were originally parameterized by fitting to data from SBDART for a range of atmospheric profiles. Vallis et al. (2018) includes the CO_2 absorption in each band and changes the functional form of the non-window optical depth. The upward flux of shortwave radiation is assumed to be transparent.

Following Eq. (6) in Vallis et al. (2018), the optical thickness in the shortwave band ($< 4 \mu\text{m}$) is parameterized as

$$\frac{\partial \tau_{\text{SW}}}{\partial \sigma} = a_{\text{SW}} + b_{\text{SW}}(\tau_{\text{SW}}) q_v + c_{\text{SW}} \ln \left(\frac{C_{\text{CO}_2}}{360} \right), \quad (4.63)$$

where the humidity coefficient depends on the accumulated optical thickness as,

$$\ln b_{\text{SW}}(\tau_{\text{SW}}) = \frac{0.01887}{\tau_{\text{SW}} + 0.009522} + \frac{1.603}{(\tau_{\text{SW}} + 0.5194)^2}. \quad (4.64)$$

This dependence approximately represents the saturation of water-vapor absorption in the shortwave spectrum.

The longwave optical thickness is divided into a non-window band in 8-14 μm and an atmospheric window band in other long-wave wavelengths ($> 4 \mu\text{m}$). Following Eq. (7) in Vallis et al. (2018), the non-window component is parameterized as

$$\frac{\partial \tau_{\text{LW}}^{\text{nw}}}{\partial \sigma} = a_{\text{LW}} + b_{\text{LW}} \ln(1 + c_{\text{LW}} q_v) + d_{\text{LW}} \ln \left(\frac{C_{\text{CO}_2}}{360 \text{ ppm}} \right), \quad (4.65)$$

while the atmospheric window band is expressed as

$$\frac{\partial \tau_{\text{LW}}^{\text{win}}}{\partial \sigma} = a_{\text{win}} + b_{\text{win}} q_v + c_{\text{win}} q_v^2 + d_{\text{win}} \ln \left(\frac{C_{\text{CO}_2}}{360 \text{ ppm}} \right). \quad (4.66)$$

The total longwave flux is obtained as the weighted sum of the two bands. Fixed weighting factors R_{nw} and R_{win} are prescribed for the non-window and window components, respectively.

The default parameter values adopted in the Isca implementation of Vallis et al. Vallis et al. (2018) are

$$(a_{\text{SW}}, c_{\text{SW}}) = (0.0596, 0.0029), \quad (4.67)$$

$$(a_{\text{LW}}, b_{\text{LW}}, c_{\text{LW}}, d_{\text{LW}}) = (0.1000, 23.8, 254.0, 0.2023), \quad (4.68)$$

$$(a_{\text{win}}, b_{\text{win}}, c_{\text{win}}, d_{\text{win}}) = (0.2150, 147.11, 10814.0, 0.0954). \quad (4.69)$$

The partitioning factors for the longwave radiation are $R_{\text{nw}} = 0.6268$ and $R_{\text{win}} = 0.3732$.

4.5.4 Common formulation

The gray, one-band, and two-band radiation schemes share the same numerical procedure for evaluating radiative fluxes. The parameterizations described in the preceding sections determine the optical-thickness gradient $\partial \tau / \partial \sigma$, whereas the radiative transfer equations are solved using a common discretization and integration procedure.

Radiative transfer is solved independently for each vertical column. The vertical grid used for the radiation calculation coincides with the DG nodal grid. For each interval between adjacent DG nodes, the optical thickness for each band is approximated by

$$\Delta \tau = \left(\frac{\partial \tau}{\partial \sigma} \right) \Delta \sigma, \quad (4.70)$$

where $\Delta \sigma = \Delta p / P_0$, and the optical-thickness gradient is evaluated using the arithmetic mean of the thermodynamic variables at the two adjacent nodes. For the one-band solar radiation scheme, the coefficient b_{SW} depends on the accumulated shortwave optical thickness and is therefore evaluated sequentially from the model top downward.

Radiative fluxes are calculated using the two-stream approximation without scattering. Assuming that the optical thickness and the Planck source function are constant within each vertical interval, the analytical solution of the radiative-transfer equation gives the upward and downward longwave fluxes as

$$F_{k+1}^{\uparrow} = F_k^{\uparrow} e^{-D\Delta\tau} + \mathcal{B} (1 - e^{-D\Delta\tau}), \quad (4.71)$$

$$F_k^{\downarrow} = F_{k+1}^{\downarrow} e^{-D\Delta\tau} + \mathcal{B} (1 - e^{-D\Delta\tau}), \quad (4.72)$$

where k is the vertical index, D is the diffusivity factor, and $\mathcal{B} = \sigma_{\text{SB}} T^4$ is the blackbody emission. When the radiation schemes described in Secs. 4.5.2 and 4.5.3 are used, we set $D = 1$. For the two-band scheme, the above calculation is applied independently to the non-window and window bands, and the total longwave flux is obtained as their weighted sum.

The downward shortwave flux is evaluated only by atmospheric attenuation,

$$F_k^{\downarrow} = F_{k+1}^{\downarrow} e^{-\Delta\tau}, \quad (4.73)$$

whereas the upward shortwave flux at the surface is determined by the prescribed surface albedo. In the present implementation, the atmosphere is assumed to be transparent to upward shortwave radiation.

Finally, the net radiative flux,

$$F_{\text{net}} = F^{\uparrow} - F^{\downarrow}, \quad (4.74)$$

is differentiated in the vertical direction using the DG derivative operator to obtain the radiative heating rate.

4.6 Surface flux

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Surface fluxes represent the exchange of momentum, heat, and moisture between the surface and the lowest model level of the atmosphere. SCALE-DG provides two surface-flux parameterizations: a constant-flux and a simple bulk-flux scheme. In the DG framework, the surface fluxes are imposed as lower-boundary conditions at the model bottom.

4.6.1 Constant flux

This option prescribes constant surface heat and moisture fluxes independently of the atmospheric state. On the other hand, the surface momentum flux is calculated using either a prescribed bulk coefficient for momentum C_m or a prescribed friction velocity u_* . The surface momentum flux is written as

$$F_{\rho\mathbf{u},\text{sfc}} = -C_m |\tilde{\mathbf{u}}| \tilde{\rho} \tilde{\mathbf{u}} \quad (4.75)$$

where the tilde denotes the value at the lowest model level of the atmosphere. When the friction velocity is prescribed, C_m is diagnosed as

$$C_m = \left(\frac{u_*}{|\tilde{\mathbf{u}}|} \right)^2. \quad (4.76)$$

In SCALE-DG, the constant-flux scheme is calculated using the SCALE library module `scale_atmos_phy_sf_const.F90`. The minimum values of the wind speed and C_m can be modified from the default values (i.e., zero) to avoid numerical instability.

4.6.2 Simple bulk flux scheme

This option calculates the surface fluxes based on constant values of the bulk drag coefficient C_m , the heat-transfer coefficient C_h , and the moisture-transfer coefficient C_e . The surface momentum, sensible heat, and latent heat fluxes are calculated as

$$F_{\rho\mathbf{u},\text{sfc}} = -C_m |\tilde{\mathbf{u}}| \tilde{\rho} \tilde{\mathbf{u}}, \quad (4.77)$$

$$F_{\text{SH},\text{sfc}} = C_h |\tilde{\mathbf{u}}| \tilde{\rho} (T_{\text{sfc}} - \tilde{T}), \quad (4.78)$$

$$F_{\text{LH},\text{sfc}} = LC_e |\tilde{\mathbf{u}}| \tilde{\rho} (q_{\text{sfc}} - \tilde{q}). \quad (4.79)$$

where the tildes denote the values at the lowest model level of the atmosphere, and T_{sfc} and q_{sfc} are the surface temperature and surface specific humidity, respectively. The surface specific humidity is calculated from the surface temperature and the surface pressure using the Clausius-Clapeyron equation as

$$q_{\text{sfc}} = \frac{\epsilon p_{\text{sat},\text{sfc}}}{p_{\text{sfc}} - (1 - \epsilon) p_{\text{sat},\text{sfc}}}, \quad (4.80)$$

where ϵ is the ratio of the molecular weight of water vapor to that of dry air, and $p_{\text{sat},\text{sfc}}$ is the saturation vapor pressure at T_{sfc} .

The minimum values of the wind speed can be modified from the default values (i.e., zero) to avoid numerical instability.

4.7 Ocean model

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4.7.1 A single-layer slab ocean model

The current version of SCALE-DG provides a simple single-layer slab ocean model. The model represents only the thermodynamics of the upper ocean and predicts the evolution of the sea surface temperature (SST). The slab ocean is treated independently at each DG node, without horizontal ocean circulation or vertical mixing below the mixed layer. This model is intended for idealized experiments and for studies in which ocean dynamics play only a secondary role. This model treats only the thermodynamics of a single-layer ocean and predicts the evolution of the sea surface temperature.

The governing equation is described using the internal energy of ocean as

$$\begin{aligned} \frac{\partial(C_o e_{\text{in},o})}{\partial t} = & R_{\text{SW},\text{dn}} - R_{\text{SW},\text{up}} + R_{\text{LW},\text{dn}} - R_{\text{LW},\text{up}} - F_{\text{SH}} - F_{\text{LH}} \\ & + e_{\text{in},\text{precip}} + Q_{\text{ext}}, \end{aligned} \quad (4.81)$$

where C_o is the heat capacity of the slab ocean and $e_{\text{in},o}$ is its internal energy. For the surface radiation fluxes $R_{*,*}$, the subscripts SW, LW, dn, and up denote shortwave radiation, longwave radiation, downward, and upward fluxes, respectively. F_{SH} and F_{LH} are the sensible and latent heat fluxes exchanged between the ocean and the atmosphere. $e_{\text{in},\text{precip}}$ represents the internal energy supplied by precipitation. Finally, Q_{ext} denotes an externally prescribed heat source, such as an idealized ocean heat transport or a prescribed Q-flux.

The heat capacity of the slab ocean is assumed to be constant and is calculated as

$$C_o = \rho_o c_{p,o} h_o, \quad (4.82)$$

where ρ_o and $c_{p,o}$ are the density and specific heat capacity of seawater, respectively, and h_o is the prescribed depth of the slab ocean. Increasing h_o increases the thermal inertia of the ocean and therefore lengthens the response time of the SST to atmospheric forcing.

The downward radiative fluxes, $R_{\text{SW},\text{dn}}$ and $R_{\text{LW},\text{dn}}$, are provided by the radiation schemes described in Sec. 4.5. The upward shortwave radiation is calculated using the prescribed ocean surface albedo α_o as

$$R_{\text{SW},\text{up}} = \alpha_o R_{\text{SW},\text{dn}}, \quad (4.83)$$

whereas the upward longwave radiation is diagnosed from the SST according to the Stefan–Boltzmann law,

$$R_{\text{LW},\text{up}} = \sigma_{\text{SB}} T_s^4, \quad (4.84)$$

where σ_{SB} is the Stefan–Boltzmann constant and T_s is the sea surface temperature diagnosed from the internal energy of the slab ocean. The sensible and latent heat fluxes are supplied by the surface-flux scheme. In the current implementation, the simple bulk-flux scheme described in Sec. 4.6.2 is used.

Because the slab ocean possesses a much larger heat capacity than the atmosphere, its characteristic evolution time is much longer than the atmospheric time step. Therefore, the governing equation Eq. (4.81) is integrated using the forward Euler method.

A The detail numerics

A.1 Runge–Kutta scheme

This section provides Butcher tables and coefficients with Shu–Osher form of Runge–Kutta schemes.

Table A.1: Double Butcher table for a third-order IMEX RK scheme proposed by Kennedy and Carpenter (2003).

c_s	$a_{ss'}$			
0	0	0	0	0
$\frac{1767732205903}{2027836641118}$	$\frac{1767732205903}{2027836641118}$	0	0	0
$\frac{3}{5}$	$\frac{5535828885825}{10492691773637}$	$\frac{788022342437}{10882634858940}$	0	0
1	$\frac{6485989280629}{16251701735622}$	$\frac{4246266847089}{9704473918619}$	$\frac{10755448449292}{10357097424841}$	0
b_s	$\frac{1471266399579}{7840856788654}$	$\frac{4482444167858}{7529755066697}$	$\frac{11266239266428}{11593286722821}$	$\frac{1767732205903}{4055673282236}$
$\tilde{c}_s$	$\tilde{a}_{ss'}$			
0	0	0	0	0
$\frac{1767732205903}{2027836641118}$	$\frac{1767732205903}{4055673282236}$	$\frac{1767732205903}{4055673282236}$	0	0
$\frac{3}{5}$	$\frac{2746238789719}{10658868560708}$	$\frac{640167445237}{6845629431997}$	$\frac{1767732205903}{4055673282236}$	0
1	$\frac{1471266399579}{7840856788654}$	$\frac{4482444167858}{7529755066697}$	$\frac{1767732205903}{11593286722821}$	$\frac{1767732205903}{4055673282236}$
$\tilde{b}_s$	$\frac{1471266399579}{7840856788654}$	$\frac{4482444167858}{7529755066697}$	$\frac{11266239266428}{11593286722821}$	$\frac{1767732205903}{4055673282236}$

Table A.2: Butcher table and Shu-Osher form for a third-order fully explicit SSP RK scheme with three stages

c_s	$a_{ss'}$		
0			
1	1		
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{4}$	
b_s	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$

$\alpha_{ss'}$	$\beta_{ss'}$		
1	1		
$\frac{3}{4}$	$\frac{1}{4}$		
$\frac{1}{3}$	0	$\frac{2}{3}$	

Table A.3: Butcher table and Shu-Osher form for a third-order fully explicit SSP RK scheme with four stages

c_s	$a_{ss'}$			
0				
$\frac{1}{2}$	$\frac{1}{2}$			
1	$\frac{1}{2}$	$\frac{1}{2}$		
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	
b_s	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{2}$

$\alpha_{ss'}$	$\beta_{ss'}$			
1	1			
0	1			
$\frac{2}{3}$	0	$\frac{1}{3}$		
0	0	0	1	

Table A.4: Shu-Osher form for a third-order and five-stage fully explicit SSP RK scheme with $2N_2^*$ storage

$\alpha_{ss'}$					$\beta_{ss'}$					
1					0.465388589249323					
0	1				0	0.465388589249323				
0.682342861037239	0	0.317657138962761			0	0	0.124745797313998			
0	0	0	1		0	0	0	0.465388589249323		
0.045230974482400	0	0	0	0.954769025517600	0	0	0	0	0.154263303748666	

Table A.5: Butcher table for a fourth-order fully explicit RK scheme with ten stages proposed by Ketcheson (2008).

c_s	$a_{ss'}$									
0										
$\frac{1}{6}$	$\frac{1}{6}$									
$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{6}$								
$\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$							
$\frac{3}{4}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$						
$\frac{7}{8}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$					
$\frac{9}{10}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{6}$				
$\frac{11}{12}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{6}$	$\frac{1}{6}$			
$\frac{13}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$		
$\frac{14}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	
1	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{15}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{10}$
b_s	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$	$\frac{1}{10}$

$\alpha_{ss'}$					$\beta_{ss'}$					
1					$\frac{1}{6}$					
0	1				0	$\frac{1}{6}$				
0	0	1			0	0	$\frac{1}{6}$			
0	0	0	1		0	0	0	$\frac{1}{6}$		
$\frac{3}{5}$	0	0	0	$\frac{2}{5}$	0	0	0	0	$\frac{1}{15}$	
0	0	0	0	0	1				$\frac{1}{6}$	
0	0	0	0	0	0	1			$\frac{1}{6}$	
0	0	0	0	0	0	0	1		$\frac{1}{6}$	
0	0	0	0	0	0	0	0	1	$\frac{1}{6}$	
0	0	0	0	0	0	0	0	0	0	$\frac{1}{10}$

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