Astrophysical tides and planet–star interactions

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Outline

● Introduction

● Linear tides in uniformly rotating unstratified fluids
  (with Doug Lin, UC Santa Cruz / KIAA, Beijing)

● Breaking internal gravity waves at the centre of a star
  (with Adrian Barker, DAMTP / Northwestern)

● Effective viscosity of turbulent convection and other flows
  (with Geoffroy Lesur, DAMTP / IPAG, Grenoble)

● Conclusions
Introduction
Tidal interactions

solar-type binary star, \( P \approx 10 \) days

jovian satellite, \( P \approx 3 \) days

hot Jupiter, \( P \approx 3 \) days

synchronization – circularization – orbital migration – tidal heating
Algol Binaries

Accretion annulus
Hot impact region
Gas stream
Starspot
Coronal loops
Corona
Size of the Sun
Hot Jupiters

- Gravitational, thermal and magnetic interactions

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Tidal forcing

- Tidal potential experienced by body 1

\[ \Psi = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} \Psi_{l,m}(t) \left( \frac{r}{R_1} \right)^l Y_{l,m}(\theta, \phi) \]
Tidal forcing

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\]

\[
\Psi_{l,m} = -C_{l,m} \frac{GM_2}{R_1} \left( \frac{R}{R_1} \right)^{-(l+1)} e^{-im\Phi}
\]
Tidal forcing

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Tidal forcing

- Tidal potential experienced by body 1

\[
Ψ = \sum_{l=2}^{\infty} \sum_{m=-l}^{l} Ψ_{l,m}(t) \left( \frac{r}{R_1} \right)^l Y_{l,m}(θ, φ)
\]

\[
Ψ_{l,m} = -C_{l,m} \frac{G M_2}{R_1} \left( \frac{R}{R_1} \right)^{-(l+1)} e^{-imΦ}
\]

|\(Ψ_{2,m}\)|
\[
e = 0.1
\]

|\(Ψ_{2,m}\)|
\[
e = 0.5
\]
Fourier analysis

\[ |\tilde{\Psi}_{2,0}| \]

\[ |\tilde{\Psi}_{2,2}| \]

\[ e = 0.1 \]

\[ e = 0.5 \]

frequency

frequency
Love number and “tidal Q”

- Consider each potential component experienced by body 1

\[ \Psi = \tilde{\Psi}_{l,m} \left( \frac{r}{R_1} \right)^l Y_{l,m}(\theta, \phi) e^{-i\omega t} \]
Love number and “tidal Q”

- Consider each potential component experienced by body 1

\[ \Psi = \tilde{\Psi}_{l,m} \left( \frac{r}{R_1} \right)^l Y_{l,m}(\theta, \phi) e^{-i\omega t} \]

- Body 1 is deformed and generates an external potential

\[ \Phi' = k_{l,m}(\omega) \tilde{\Psi}_{l,m} \left( \frac{r}{R_1} \right)^{-(l+1)} Y_{l,m}(\theta, \phi) e^{-i\omega t} \]

\[ ( + \text{ orthogonal terms } ) \]

- Potential Love number (linear response function)
Love number and “tidal Q”

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( + orthogonal terms )

- Potential Love number (linear response function)

- Energy transfer to orbit \( \propto \omega \text{Im}(k_{l,m}) |\tilde{\Psi}_{l,m}|^2 \)

- Angular momentum transfer \( \propto m \text{Im}(k_{l,m}) |\tilde{\Psi}_{l,m}|^2 \)
Love number and “tidal Q”

- Consider each potential component experienced by body 1
  \[ \Psi = \tilde{\Psi}_{l,m} \left( \frac{r}{R_1} \right)^l Y_{l,m}(\theta,\phi) e^{-i\omega t} \]

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  ( + orthogonal terms )

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- Energy transfer to orbit \[ \propto \omega \text{Im}(k_{l,m}) |\tilde{\Psi}_{l,m}|^2 \]

- Angular momentum transfer \[ \propto m \text{Im}(k_{l,m}) |\tilde{\Psi}_{l,m}|^2 \]

- \[ \text{Im}(k) \approx \frac{k}{Q} \approx \frac{1}{Q'} \ll 1 \quad \text{depends on} \quad \omega, l, m \quad \text{(usually} \quad l = m = 2) \]
Analogy: forced harmonic oscillator

\[ \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f e^{-i\omega t} \]
Analogy: forced harmonic oscillator

\[
\ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f \, e^{-i\omega t}
\]

\[
x = k \frac{f}{\omega_0^2} \, e^{-i\omega t}
\]

\[
k = \left( 1 - \frac{2i\omega\gamma}{\omega_0^2} - \frac{\omega^2}{\omega_0^2} \right)^{-1}
\]

\[
\approx (1 + i Q^{-1})
\]

\[\omega, \gamma \ll \omega_0\]

\[
Q = \frac{\omega_0^2}{2\omega\gamma} = \frac{2\pi \times \text{maximum potential energy}}{\text{energy dissipated per cycle}} \gg 1
\]
Analogy: forced harmonic oscillator

\[ \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f \, e^{-i\omega t} \]

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\[ Q = \frac{\omega_0^2}{2\omega\gamma} = \frac{2\pi \times \text{maximum potential energy}}{\text{energy dissipated per cycle}} \gg 1 \]
Spin and orbital evolution

- $\Omega_1$: semimajor axis
- $\Omega_2$: eccentricity
- $M_1$: mean motion
- $n$: specific angular momentum
- $e$: eccentricity vector
Parametrized models

- Tidal timescale \( T_* = \frac{2Q'_* M_*}{9n M_p} \left( \frac{a}{R_*} \right)^5 \) etc.

- Planet pseudosynchronized: \( \Omega_p = \left( \frac{2f_4}{f_2 + f_3} \right) n \)

- Magnetic braking of star

\[
\frac{dn}{dt} = \frac{3n}{T_*} \left[ f_4 - \left( \frac{f_2 + f_3}{2} \right) \frac{\Omega_*}{n} \cos i \right]
\]

\[
\frac{de}{dt} = -\frac{9e}{T_*} \left[ f_1 - \frac{11}{18} f_2 \frac{\Omega_*}{n} \cos i \right] - \frac{9e}{T_p} \left[ f_1 - \frac{11}{18} \left( \frac{2f_2 f_4}{f_2 + f_3} \right) \right]
\]

\[
\frac{d\Omega_*}{dt} = \frac{\mu n a^2}{I_* T_*} \left[ f_4 \cos i - \frac{\Omega_*}{2n} (f_2 + f_3 \cos^2 i) \right] - \alpha \Omega_*^3
\]

\[
\frac{di}{dt} = -\frac{\Omega_*}{2n T_*} f_2 \sin i - \frac{\mu n a^2}{I_* \Omega_* T_*} \left[ f_4 - \frac{\Omega_*}{2n} f_3 \cos i \right] \sin i
\]
Parametrized models

- e.g. Barker & Ogilvie (2009)

\[
\begin{align*}
1 \, M_\odot & \quad Q'_* = 10^6 \\
1 \, M_J & \quad Q'_p = 10^6 \\
P_{*0} & = 1 \text{ day} \\
\alpha & = 1.5 \times 10^{-14} \text{ yr}
\end{align*}
\]
Initial stellar obliquity 0°
Initial stellar obliquity 45°
Initial stellar obliquity 90°
4000 Myr

- orbit equation
- eccentricity
- orbital period / days
No tide in planet
Timescales (for small $e$)

- Circularization (using tide in planet)

$$3 Q'_p \frac{M_p}{M_J} \left( \frac{R_p}{R_J} \right)^{-5} \left( \frac{M_*}{M_\odot} \right)^{2/3} \left( \frac{P}{\text{day}} \right)^{13/3} \text{ yr}$$

- Orbital decay (inspiral)

$$0.02 Q'_* \frac{M_*}{M_p} \left( \frac{\bar{\rho}}{\bar{\rho}_\odot} \right)^{5/3} \left( \frac{P}{\text{day}} \right)^{13/3} \text{ yr}$$

- Spin-orbit alignment

$$0.07 Q'_* \frac{M_*}{M_p} \left( \frac{\bar{\rho}}{\bar{\rho}_\odot} \right)^{5/3} \left( \frac{P}{\text{day}} \right)^{13/3} \text{ yr}$$

(but reduced if orbit has more angular momentum than stellar spin)
Tidal dissipation in rotating stars and giant planets

J-P Zahn’s categorization:

- "Equilibrium tide"
  Dissipation associated with large-scale tidal bulge

\[ r^2 Y_{2,m}(\theta, \phi) e^{-i\omega t} \]

- "Dynamical tide"
  Dissipation associated with low-frequency waves
Analogy: forced harmonic oscillator

\[ \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f \ e^{-i\omega t} \]

\[ x = k \ \frac{f}{\omega_0^2} \ e^{-i\omega t} \]

\[ k = \left( 1 - \frac{2i\omega\gamma}{\omega_0^2} - \frac{\omega^2}{\omega_0^2} \right)^{-1} \]

\[ \approx \ (1 + iQ^{-1}) \]

\[ [\omega, \gamma \ll \omega_0] \]

\[ Q = \frac{\omega_0^2}{2\omega\gamma} = \frac{2\pi \times \text{maximum potential energy}}{\text{energy dissipated per cycle}} \approx 1 \]
Analogy: forced harmonic oscillator

\[ \ddot{x} + 2\gamma \dot{x} + \omega_0^2 x = f \, e^{-i\omega t} \]

\[ x = k \frac{f}{\omega_0^2} e^{-i\omega t} \]

\[ k = \left(1 - \frac{2i\omega\gamma}{\omega_0^2} - \frac{\omega^2}{\omega_0^2}\right)^{-1} \approx (1 + iQ^{-1}) \]

\[ \omega_0 \ll \omega \]

\[ Q = \frac{\omega_0^2}{2\omega\gamma} = \frac{2\pi \times \text{maximum potential energy}}{\text{energy dissipated per cycle}} \gg 1 \]
Normal modes and tidal overlap integrals

- Assume the body has a complete orthogonal set of ideal oscillation modes
- Calculate weak damping rates
- Project tidal forcing on to normal modes
- Each responds as a forced damped harmonic oscillator
Normal modes and tidal overlap integrals

- Assume the body has a complete orthogonal set of ideal oscillation modes
- Calculate weak damping rates
- Project tidal forcing on to normal modes
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Cowling (1941): non-rotating star
- f, p and g modes
- most frequencies too high
- high-order g modes
Normal modes observed in the Sun
Oscillations with rotation

- No complete theory!

- Helioseismology:
  fast modes
  $\Omega \rightarrow$ small correction

  Thompson et al. (1996)

- Tides:
  slow modes
  $\Omega \rightarrow$ radical alteration
  and new modes / waves
Normal modes and tidal overlap integrals

- Assume the body has a complete orthogonal set of ideal oscillation modes
- Calculate weak damping rates
- Project tidal forcing on to normal modes
- Each responds as a forced damped harmonic oscillator

Rotating stars and planets:
Approach may fail if waves don’t reflect to form standing modes, or if normal modes can’t be defined because of singularities

Christensen-Dalsgaard (2003)
Tidal forcing problem

Viscous uniformly rotating fluid

Tidal potential $\Psi$ and linear response proportional to $e^{-i\omega t}$

\[
\begin{align*}
-i\omega u_i + 2\epsilon_{ijk}\Omega_j u_k &= -\partial_i \Phi' - \partial_i \Psi - \frac{1}{\rho} \partial_i p' + \frac{\rho'}{\rho^2} \partial_i p + \frac{1}{\rho} \partial_j T_{ij} \\
-i\omega \rho' + u_i \partial_i \rho &= -\rho \partial_i u_i \\
-i\omega p' + u_i \partial_i p &= -\Gamma_1 p \partial_i u_i \\
T_{ij} &= 2\mu S_{ij} + \mu_b (\partial_k u_k) \delta_{ij} \\
S_{ij} &= \partial_i u_j + \partial_j u_i - \frac{2}{3} (\partial_k u_k) \delta_{ij} \\
\partial_{ii} \Phi' &= 4\pi G \rho'
\end{align*}
\]

Given $\Psi = \left(\frac{r}{R}\right)^l Y_{l,m}(\theta,\phi)$

find $\Phi' = k_{l,m} \left(\frac{r}{R}\right)^{-(l+1)} Y_{l,m}(\theta,\phi) + \cdots \quad (r > R)$
Tidal forcing problem

Energy dissipation rate

\[ D = \frac{1}{2} \int (2\mu S_{ij}^* S_{ij} + \mu_b |\partial_i u_i|^2) \, dV \]

Energy input rate

\[ -\frac{1}{2} \omega \text{Im} \int \rho' \Psi^* \, dV = \text{Im}(k_{l,m}) \frac{(2l + 1)R}{8\pi G} = D \]

Tidal torque

\[ T = \frac{m}{\omega} D \]

Complications:

differential rotation, thermal diffusion, convection, magnetic fields, nonlinearity, ...
rigidity and $Q$ the specific dissipation function

$$Q = \frac{2\pi E^*}{\int (dE/dt)dt},$$

where $E^*$ is the peak energy stored in the system during a cycle and $\int (dE/dt)dt$ is the energy dissipated over a complete cycle. $Q$ will in general vary with the frequency and amplitude of the tide and the size of the sphere in addition to its composition.

Cycle per second to one cycle per year (8). In this case $\epsilon_1$ must be the dominant term and the sign of $(de/dt)_p$ is the same as the sign of $2\omega - 3n$. While this constant behaviour of $Q$ with frequency may not be true for all planets (especially not the major ones) it is still likely that the $\epsilon_1$ term is dominant because of its relatively large coefficient. If this $\epsilon_1$ term is dominant, we have $(de/dt)_p > 0$ for all satellites. Small amplitude will have a phase lag which increases when its peak is reinforcing the peak of the tide of major amplitude. This non-linear behaviour cannot be treated in detail since very little is known about the response of the planets to tidal forces, except for the Earth. In our discussions we shall use the language of linear tidal theory, but we must keep in mind that our numbers are really only parametric fits to a non-linear problem.
Nonlinearity of tides in fluid bodies

- Equilibrium tidal amplitude

\[ \frac{\xi}{R_1} \sim \frac{M_2}{M_1} \left( \frac{R_1}{a} \right)^3 \sim 1 \quad \text{for} \quad R_1 \sim \left( \frac{M_1}{M_2} \right)^{1/3} a \]

- Nonlinear breakdown through secondary instabilities when

\[ \frac{\xi}{R_1} \sim \left( \frac{\nu}{R_1^2 \omega} \right)^{1/2} \ll 1 \quad \text{or} \quad \frac{\xi}{R_1} \sim \frac{\nu}{R_1^2 \omega} ? \]

- Internal wave nonlinearity

\[ \sim \frac{\xi_{\text{wave}}}{\lambda} \]
Algol Binaries

- Accretion annulus
- Hot impact region
- Gas stream
- Starspot
- Corona loops
- Corona

Size of the Sun
Nonlinearity of tides in fluid bodies

- Equilibrium tidal amplitude

\[ \frac{\xi}{R_1} \sim \frac{M_2}{M_1} \left( \frac{R_1}{a} \right)^3 \sim 1 \quad \text{for} \quad R_1 \sim \left( \frac{M_1}{M_2} \right)^{1/3} a \]

- Nonlinear breakdown through secondary instabilities when

\[ \frac{\xi}{R_1} \sim \left( \frac{\nu}{R_1^2 \omega} \right)^{1/2} \ll 1 \quad \text{or} \quad \frac{\xi}{R_1} \sim \frac{\nu}{R_1^2 \omega} \]

- Internal wave nonlinearity

\[ \sim \frac{\xi_{\text{wave}}}{\lambda} \]
Galilean moons of Jupiter

- Io
- Europa
- Ganymede
- Callisto

2:1 period ratio
Galilean moons of Jupiter

Assembly and maintenance of Laplace resonance:

- $2 \times 10^5 < Q'_J < 8 \times 10^6$  
  (Goldreich 1965, Yoder & Peale 1981)
Naive backward tidal evolution of Galilean satellites

- $Q'_j = 10^6$

![Graph showing the tidal evolution of Galilean satellites](image-url)
Galilean moons of Jupiter

Lainey et al. (2009)

<table>
<thead>
<tr>
<th>Ref.</th>
<th>Secular mean-motion acceleration ($\dot{\eta}/n$) ($10^{-10}$ yr$^{-1}$)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Io</td>
</tr>
<tr>
<td>9</td>
<td>$+3.3 \pm 0.5$</td>
</tr>
<tr>
<td>10</td>
<td>$-0.074 \pm 0.087$</td>
</tr>
<tr>
<td>24</td>
<td>$+4.54 \pm 0.95$</td>
</tr>
<tr>
<td>25</td>
<td>$+2.27 \pm 0.70$</td>
</tr>
<tr>
<td>26</td>
<td>$+3.6 \pm 1.0$</td>
</tr>
<tr>
<td>This paper</td>
<td>$+0.14 \pm 0.01$</td>
</tr>
</tbody>
</table>

\[ Q'_{J,\text{Io,now}} = 1.4 \times 10^5 \]
Naive backward tidal evolution of inner moons of Saturn

- $Q'_S = 10^5$
Solar-type binary stars
Solar-type binary stars

Meibom & Mathieu (2005)

M35, 150 Myr

log10 (orbital period in days)

orbital eccentricity

log10 (orbital period in days)
Solar-type binary stars

Meibom & Mathieu (2005)
Solar-type binary stars

Meibom & Mathieu (2005)
Hot Jupiters

[Image of the Sun and a planet, possibly a Hot Jupiter, against a black background.]

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Short-period extrasolar planets

![Graph showing the relationship between Msin(i) [Jupiter Mass] and Orbital Period [Days].](exoplanets.org | 11/30/2011)
Short-period extrasolar planets
Short-period extrasolar planets
SOME KEY OBSERVATIONS

Very short orbital periods
  • why do planets not spiral in?

Some eccentric orbits among these
  • why are orbits not circularized?

(Mis)alignment of stellar spin and orbit
  • clue to origin of hot Jupiters?

Radii of planets
  • internal structure and heating
WASP-19 b

P = 0.789 day
ea = 0.0163 AU

M_P = 1.11 ± 0.04 M_J
R_P = 1.39 ± 0.03 R_J

= 0.56 R_H

e = 0.005 ± 0.004

λ = 5 ± 5°

M_S = 0.93 ± 0.02 M_⊙ (G8V)
R_S = 0.99 ± 0.02 R_⊙

P_S = 10.5 ± 0.2 day
t = 0.5–0.6 Gyr

Hebb & 10

_{t_e} = 0.0003 (Q'_P/10^6) Gyr
_{t_a} = 0.0076 (Q'_S/10^6) Gyr
WASP-18 b

\[ P = 0.941 \text{ day} \]
\[ a = 0.020 \text{ AU} \]
\[ M_P = 10.1 \pm 0.3 \text{ M}_J \]
\[ R_P = 1.11 \pm 0.06 \text{ R}_J \]
\[ e = 0.009 \pm 0.003 \]
\[ \lambda = 4 \pm 5^\circ \]

\[ M_S = 1.22 \pm 0.03 \text{ M}_\odot \text{ (F6V)} \]
\[ R_S = 1.22 \pm 0.06 \text{ R}_\odot \]
\[ P_S \approx 6 \text{ day} \]
\[ t = 0.5 - 1.5 \text{ Gyr} \]

\[ t_e = 0.017 \left( \frac{Q'_P}{10^6} \right) \text{ Gyr} \text{ or } 0.0019 \left( \frac{Q'_S}{10^6} \right) \text{ Gyr} \]
\[ t_a = 0.0010 \left( \frac{Q'_S}{10^6} \right) \text{ Gyr} \]

Hellier & 09

2011年12月2日金曜日
WASP-12 b

\[ P = 1.091 \text{ day} \]
\[ a = 0.023 \text{ AU} \]

\[ M_P = 1.35 \pm 0.05 \text{ M}_J \]
\[ R_P = 1.79 \pm 0.09 \text{ R}_J \]
\[ = 0.53 \text{ R}_H \]

(See Li et al. 2010)

\[ e = 0.049 \pm 0.015 \]
\[ M_S = 1.28 \pm 0.05 \text{ M}_\odot \text{ (F6V)} \]
\[ R_S = 1.63 \pm 0.08 \text{ R}_\odot \]
\[ P_S > 20 \text{ day} \]
\[ t \approx 2 \text{ Gyr} \]

\[ t_e = 0.0004 \left( \frac{Q_P}{10^6} \right) \text{ Gyr} \]
\[ t_a = 0.0050 \left( \frac{Q_S}{10^6} \right) \text{ Gyr} \]
OGLE-TR-56 b

Konacki & 03, Torres & 08

\[ P = 1.212 \text{ day} \]
\[ a = 0.024 \text{ AU} \]
\[ M_P = 1.39 \pm 0.18 \, M_J \]
\[ R_P = 1.36 \pm 0.09 \, R_J \]
\[ M_S = 1.23 \pm 0.07 \, M_\odot \]
\[ R_S = 1.36 \pm 0.09 \, R_\odot \]
\[ e = 0 \]
\[ t = 1.9 - 4.2 \, \text{Gyr} \]

\[ t_e = 0.002 \, (Q'_P/10^6) \, \text{Gyr} \]
\[ t_a = 0.013 \, (Q'_S/10^6) \, \text{Gyr} \]
HD 41004 B b

Zucker & 04

$P = 1.328 \text{ day}$

$a = 0.018 \text{ AU}$

$M_P \sin i = 18 \pm 1 \text{ M}_J$

$e = 0.08 \pm 0.01$

$M_S = 0.4 \pm 0.04 \text{ M}_\odot (\text{M2.5V})$

$t_e \approx 0.1 \left( \frac{Q'_P}{10^6} \right) \text{ Gyr} \text{ or } 0.1 \left( \frac{Q'_S}{10^6} \right) \text{ Gyr}$

$t_a \approx 0.05 \left( \frac{Q'_S}{10^6} \right) \text{ Gyr}$
GJ 436 b

\[ P = 2.644 \text{ day} \]
\[ a = 0.029 \text{ AU} \]
\[ M_P = 0.073 \pm 0.003 \text{ M}_J \]
\[ R_P = 0.377 \pm 0.009 \text{ R}_J \]
\[ t = 1 - 10 \text{ Gyr} \]
\[ e = 0.16 \pm 0.05 \]
\[ M_S = 0.45 \pm 0.01 \text{ M}_\odot \text{ (G8V)} \]
\[ R_S = 0.46 \pm 0.01 \text{ R}_\odot \]
\[ t_e = 1.2 \left( \frac{Q'_P}{10^6} \right) \text{ Gyr} \]
\[ t_a = 110 \left( \frac{Q'_S}{10^6} \right) \text{ Gyr} \]
XO-3 b

Johns-Krull & 08, Winn & 09

$P = 3.192$ day
$a = 0.048$ AU

$M_P = 13.3 \pm 0.6\ M_J$
$R_P = 1.22\ R_J$

$e = 0.288 \pm 0.004$
$\lambda = 37.3 \pm 3.7^\circ$

$M_S = 1.41 \pm 0.08\ M_\odot\ (F5V)$
$R_S = 2.13 \pm 0.21\ R_\odot$
$P_S < 3.73 \pm 0.23$ day

$te = 0.025\ (Q's/10^6)\ Gyr$
$t_a = 0.014\ (Q's/10^6)\ Gyr$
Tidal dissipation in rotating stars and giant planets

J-P Zahn’s categorization:

- “Equilibrium tide”
  Dissipation associated with large-scale tidal bulge

\[ r^2 Y_{2,m}(\theta, \phi) e^{-i\omega t} \]

- “Dynamical tide”
  Dissipation associated with low-frequency waves
Tidal Q of solar-type stars and giant planets

No simple answer!

- $Q$ (or $k_{l,m}(\omega)$) is a response function, not a simple number
- Fluid dynamical calculations are still exploratory
- Planetary interior models are uncertain
Tidal dissipation in rotating stars and giant planets

“Equilibrium tide”
- solid regions (viscoelastic, etc.)
- convective regions (turbulent “viscosity”)
- other physics (phase transitions, helium separation)
- nonlinear breakdown (elliptical instability, etc.)

“Dynamical tide”
- inertial waves in convective regions
- inertia-gravity waves in radiative regions
Elliptical instability in a deformed rotating sphere

Tidal dissipation in rotating stars and giant planets

“Equilibrium tide”

- solid regions (viscoelastic, etc.)
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“Dynamical tide”

- inertial waves in convective regions
- inertia-gravity waves in radiative regions
Low-frequency waves in rotating stars and giant planets

\[ \omega^2 = 4\Omega^2 (\hat{k} \cdot \hat{\Omega})^2 + N^2 (\hat{k} \times \hat{g})^2 \]

- Convective regions: inertial waves
  (Savonije et al.; Ogilvie & Lin; Wu; Ivanov & Papaloizou; Goodman & Lackner; Rieutord & Valdettaro)

- Radiative regions: inertia-gravity waves
  (Zahn; Savonije & Papaloizou; Goldreich & Nicholson; Savonije & Witte)

Excitation, propagation, reflection, dissipation

Eigenvalue problem for normal modes is generally non-separable and ill-posed, so modes may not exist without diffusion
Internal gravity waves

Buoyancy (Brunt-Väisälä) frequency

\[ N^2 = \frac{g}{\rho} \left( \frac{d\rho}{dz} \bigg|_{\text{ad}} - \frac{d\rho}{dz} \right) \]

\[ = g \left( \frac{1}{\Gamma_1} \frac{d \ln p}{dz} - \frac{d \ln \rho}{dz} \right) \]
Internal gravity waves

Buoyancy (Brunt-Väisälä) frequency

\[ N^2 = \frac{g}{\rho} \left( \frac{\partial \rho}{\partial z} \right)_{\text{ad}} - \frac{\partial \rho}{\partial z} \]

\[ = g \left( \frac{1}{\Gamma_1} \frac{d \ln p}{dz} - \frac{d \ln \rho}{dz} \right) \]

Plane gravity wave

\[ \omega = N \cos \theta \]

\[ \omega^2 = N^2 |\hat{k} \times \hat{g}|^2 \]
Inertial waves

Horizontal oscillation

\[
\begin{align*}
\dot{u}_x &= 2\Omega u_y \\
\dot{u}_y &= -2\Omega u_x
\end{align*}
\]

\( \omega = 2\Omega \)

(centrifugal force balanced by pressure)
Inertial waves

Horizontal oscillation

\[ \begin{align*}
\dot{u}_x &= 2\Omega u_y \\
\dot{u}_y &= -2\Omega u_x
\end{align*} \]

\( \omega = 2\Omega \)

(centrifugal force balanced by pressure)

Plane inertial wave

\[ \omega = 2\Omega \cos \theta \]

\[ \omega^2 = 4\Omega^2 (\hat{k} \cdot \hat{\Omega})^2 \]
Inertial waves

Horizontal oscillation

\[ \begin{align*}
\dot{u}_x &= 2\Omega u_y \\
\dot{u}_y &= -2\Omega u_x
\end{align*} \]

\( \omega = 2\Omega \)

(centrifugal force balanced by pressure)

Plane inertial wave

\[ \omega = 2\Omega \cos \theta \]

\[ \omega^2 = 4\Omega^2 (\hat{k} \cdot \hat{\Omega})^2 \]
Linear tides in uniformly rotating unstratified fluids
Tides in convective regions of planets and stars

- Turbulent viscosity acting on tidal bulge?
- Excitation and dissipation of inertial waves?
Linear tides in barotropic fluid bodies

- Barotropic: no stable stratification or internal gravity waves
- Low-frequency tides in slowly rotating bodies:
  \[ \omega \sim \Omega \sim \epsilon \left( \frac{GM}{R^3} \right)^{1/2}, \quad \epsilon \ll 1 \]

- Systematic theory based on expansion in powers of \( \epsilon^2 \)
- Displacement \( \xi = \xi_{nw} + \xi_w \)
- Non-wavelike part:
  response of spherical body to tidal potential neglecting Coriolis
  (easily computed but different from classical equilibrium tide)
- Wavelike part:
  residual response (inertial waves)
  known body force from Coriolis force on non-wavelike part
Inertial waves

Horizontal oscillation

\[ \begin{align*}
\dot{u}_x &= 2\Omega u_y \\
\dot{u}_y &= -2\Omega u_x
\end{align*} \]

\[ \omega = 2\Omega \]

(centrifugal force balanced by pressure)

Plane inertial wave

\[ \omega = 2\Omega \cos \theta \]

\[ \omega^2 = 4\Omega^2 (\hat{k} \cdot \hat{\Omega})^2 \]
Peculiarities of low-frequency waves

Wave frequency depends only on direction of wavevector

Group velocity *perpendicular* to wavevector

Focusing of beams at sloping boundaries

Reflection from interfaces...
Inertial waves in convective regions

Solar-type star

[Savonije & Witte 2002]
Ogilvie & Lin 2007

[Irradiated] giant planet

Ogilvie & Lin 2004
Wu 2005
Papaloizou & Ivanov 2005
Ivanov & Papaloizou 2007
Goodman & Lackner 2009
Ogilvie 2009
Rieutord & Valdettaro 2010
Inertial wave frequency range

For a uniformly rotating body, \(-2\Omega < \hat{\omega} < 2\Omega\)
Inertial waves: modes or beams?

Dense or continuous spectrum, \(-2\Omega < \hat{\omega} < 2\Omega\)

- Tidal forcing excites normal modes (Wu; Papaloizou & Ivanov)
- Tidal forcing excites narrow beams (Ogilvie & Lin; Goodman & Lackner; Rieutord & Valdettaro)
MODES / RESONANCE?

Full sphere (Bryan 1889):

- two-index set of smooth modes for each \( m \)
- discrete spectrum, dense in \((-2\Omega, 2\Omega)\)

- no resonant excitation by \( Y_2^2 \) (homogeneous body)
INERTIAL WAVES IN A SHELL

Spherical shell (Bretherton 1964, Stewartson 1972)

- no separation of variables, no smooth modes
- numerical calculations including viscosity
  (Rieutord et al. 2001)
RAY PROPAGATION

convergence to a wave attractor
Attractive simplicity: inertial waves in a box

$\Omega$

or

$\Omega$

$\omega = \Omega \sqrt{2}$
Wave attractors versus normal modes

![Graph showing total dissipation rate as a function of forcing frequency for tilted and untilted rotation axes.](graph.png)
Attractor asymptotics

Ogilvie (2005)

\[ i \frac{\partial^2 \psi}{\partial x \partial y} - \epsilon \nabla^2 \psi = f \]

![Graph showing total dissipation vs friction coefficient](image)

2011年12月2日金曜日
Attractor asymptotics

numerical

asymptotic
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.01

\[ \text{Ek} = \frac{\nu}{2\Omega R^2} \]

\begin{align*}
\text{Ek} &= 10^{-3} \\
\text{Ek} &= 10^{-4} \\
\text{Ek} &= 10^{-5} \\
\text{Ek} &= 10^{-6}
\end{align*}
Inertial waves: modes or beams?

Dense or continuous spectrum, $-2\Omega < \hat{\omega} < 2\Omega$

- Tidal forcing excites normal modes (Wu; Papaloizou & Ivanov)
- Tidal forcing excites narrow beams (Ogilvie & Lin; Goodman & Lackner; Rieutord & Valdettaro)
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\[ \text{Ek} = \frac{\nu}{2\Omega R^2} \]

\( \text{Ek} = 10^{-6} \)

\( \text{Ek} = 10^{-5} \)

\( \text{Ek} = 10^{-4} \)

\( \text{Ek} = 10^{-3} \)
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.1
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.2
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.3
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.4
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.5
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.6
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.7
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.8
Responses of spheres and shells

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core, fractional radius 0.9

![Graph showing log(dissipation/\omega^2) vs \omega/\Omega]
critical latitude

singularity
wave attractors
Mixed metaphors
Mixed metaphors

wave attractor ⇔ black hole [M. Rieutord]
Mixed metaphors

wave attractor $\Leftrightarrow$ black hole [M. Rieutord]
critical latitude $\Leftrightarrow$ Hawking radiation [J. Goodman]
(a) Definition sketch

$U_0 \cos (\omega_0 t)$

(b) Tidal conversion by a ridge

$Nz/\omega L$
Dependence on core size

![Graph showing the dependence of frequency-averaged dissipation above viscous baseline on fractional core size. The x-axis represents fractional core size ranging from 0.01 to 1.00, and the y-axis represents frequency-averaged dissipation on a logarithmic scale from $10^{-9}$ to $10^{-6}$. The graph includes a curve that shows an increasing trend as the fractional core size increases.]
Dependence on core size

\[ \propto \text{fifth power of core size} \]

(from overlap with (quasi) normal modes (cf. Wu; Pap. & Ivanov))

(from viscous baseline (cf. Goodman & Lackner))
Rigid versus fluid core

Idealized problem: isentropic rotating fluid in spherical geometry

- Rigid core
Rigid versus fluid core

Idealized problem: isentropic rotating fluid in spherical geometry
- Fluid core, density jump by factor 2
Responses of spheres and shells

- Full spheres with smooth density profiles support normal modes.
- Some tidal overlap with normal modes occurs, leading to resonant peaks in the response, if the density is non-uniform.
- Presence of a core and/or density jumps enhances tidal response but concepts of normal modes and resonance are less relevant.
- Enhanced dissipation for tidal frequencies (in rotating frame) $-2\Omega < \omega < 2\Omega$ relevant for synchronization and circularization.
- Frequency-averaged $Q$ strongly dependent on internal structure but not on viscosity; for intermediate core sizes,

\[
\left\langle \frac{1}{Q'} \right\rangle \omega \approx \left( \frac{R_c}{R_p} \right)^5 \left( \frac{\Omega}{\Omega_{\text{dyn}}} \right)^2
\]

- Strong frequency dependence in cases of low viscosity.
inertial-wave response of convective zone

tidal frequency equal to spin frequency

relevant to binary circularization but not planet hosts
Inertial waves in a solar-type star

Ogilvie & Lin (2007)
- solar model, but spin period 10 days
- dissipation in convective zone only

Goldreich & Nicholson viscosity
tidal period 0.5 d

tidal period 10 d
Impulsive forcing of inertial waves

- Consider inertial waves driven by body force $\propto \delta(t)$
- Impulsive response is smooth and readily computable (ODEs)
- Will subsequently resolve into complicated pattern of inertial waves which may dissipate through linear or nonlinear processes
- (Kinetic) energy of impulsive response gives a broad frequency average of the response function
- Equivalent to frequency-average of tidal time lag, or $\frac{1}{\omega Q}$
- Robust result, dependent only on gross structure
- Also more directly applicable to highly eccentric orbits
Impulsive energy transfer / frequency-averaged dissipation

- Homogeneous fluid body with rigid core
  - Sectoral harmonics \( m = l \)
    \[
    \hat{E} \propto \frac{\alpha^{2l+1}}{1 - \alpha^{2l+1}} \quad \alpha = \frac{r_{\text{in}}}{R}
    \]
  - Tesseral harmonics \( m < l \)
    \[
    \hat{E} \propto \frac{1}{1 - \alpha^{2l+1}} \quad (+ \text{term as above})
    \]
    but beware trivial inertial modes with \( l = 2 \)
Impulsive energy transfer / frequency-averaged dissipation

- Homogeneous fluid body with rigid core
  - Sectoral harmonics $m = l$
    \[
    \hat{E} \propto \frac{\alpha^{2l+1}}{1 - \alpha^{2l+1}} \quad \alpha = \frac{r_{\text{in}}}{R}
    \]
  - Tesseral harmonics $m < l$
    \[
    \hat{E} \propto \frac{1}{1 - \alpha^{2l+1}} \quad (+ \text{term as above})
    \]
    but beware trivial inertial modes with $l = 2$

- Two homogeneous fluids
  - Similar but weaker result
  - Strengthened if densities differ greatly
Impulsive energy transfer / frequency-averaged dissipation

- Polytrope with rigid core
  \[ p \propto \rho^{1+1/n} \]

\[ l = m = 2 \]

\[ r^5 \]

\[ n = 0 \text{ (homogeneous)} \]

\[ n = 0.1 \]

\[ n = 0.3 \]

\[ n = 3 \]
Processes affecting inertial waves at small scales

Viscous dissipation
Effective viscous dissipation by turbulent convection
Deflection by meridional circulation
Deflection (scattering) by turbulent convection (random medium)
Interaction with magnetic fields, and Ohmic dissipation
Wave breaking / parametric instability
Imperfect reflection from boundaries / interfaces
Viscous and Ohmic dissipation

Local dispersion relation in incompressible MHD

\[
(\omega_\nu \omega_\eta - \omega_a^2)^2 = 4(\hat{k} \cdot \Omega)^2 \omega_\eta^2
\]

\[
\omega_\nu = \omega - k \cdot u + i\nu k^2
\]

\[
\omega_\eta = \omega - k \cdot u + i\eta k^2
\]

\[
\omega_a = k \cdot v_a
\]

\[
v_a = (\mu_0 \rho)^{-1/2} B
\]

\[
\eta = \frac{1}{\mu_0 \sigma}
\]
Viscous and Ohmic dissipation

\[(\omega_\nu \omega_\eta - \omega_a^2)^2 = 4(\hat{k} \cdot \boldsymbol{\Omega})^2 \omega_\eta^2\]

Magnetic Prandtl number \quad P_m = \frac{\nu}{\eta} \ll 1

Elsasser number \quad \Lambda = \frac{v_a^2}{2\Omega \eta} = O(1)

Magnetic coupling scale \quad k_a = \frac{2\Omega}{v_a}

Resistive scale \quad k_\eta = \left(\frac{2\Omega}{\eta}\right)^{1/2} = \Lambda^{1/2} k_a

Viscous scale \quad k_\nu = \left(\frac{2\Omega}{\nu}\right)^{1/2} = P_m^{-1/2} k_\eta
Viscous and Ohmic dissipation

\[ P_m = 10^{-6} \]

\[ \Lambda = 1 \]
Breaking internal gravity waves at the centre of a star
Inertia-gravity waves in radiative regions

Solar-type star

Goodman & Dickson 1998
Terquem et al. 1998
Savonije & Witte 2002
Ogilvie & Lin 2007
Barker & Ogilvie 2010

Irradiated giant planet

[Ioannou & Lindzen 1993]
Lubow et al. 1997
Ogilvie & Lin 2004
[Gu & Ogilvie 2009]
[Arras & Socrates 2010]
Inertia-gravity waves in radiative regions

\[ \omega^2 > N^2 \]

\[ N^2 \approx 0 \]

\[ \omega^2 < N^2 \]

\[ N^2 \uparrow \]

Goldreich & Nicholson 1989
Inertia-gravity waves: resonant modes or breaking waves?
Inertia-gravity waves in radiative regions

Savonije & Witte 2002

- linear tidal response of 1-solar mass star
- realistic stellar model and evolution
- Coriolis force (traditional approximation)
- radiative diffusion
- turbulent viscosity [large?]
Inertia-gravity waves in radiative regions (star)

Savonije & Witte 2002 (cf. Terquem et al. 1998)

- resonant excitation of normal modes

\[ Q' \approx 10^7 \]

[large turbulent viscosity?]
Inertia-gravity waves in radiative regions (star)


- assumes waves do not reflect from stellar centre
Near stellar centre:

\[ \rho = \rho_0 + \rho_2 r^2 + \cdots \]
\[ p = p_0 + p_2 r^2 + \cdots \]
\[ g = g_1 r + g_3 r^3 + \cdots \]

Brunt-Väisälä frequency:

\[ N^2 = g \left( \frac{1}{\Gamma_1} \frac{d \ln p}{dr} - \frac{d \ln \rho}{dr} \right) \]
\[ N = N_1 r + N_3 r^3 + \cdots \]

\( N_1 \) generally increases with stellar mass and age

launch site

(\( B_\phi \) may interfere)
$N_1$ versus age for the Sun
BREAKING GRAVITY WAVES

Quasi-Boussinesq system:

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u = -\nabla p + b r
\]

\[
\frac{\partial b}{\partial t} + u \cdot \nabla b + N_1^2 u \cdot r = 0
\]

\[
\nabla \cdot u = 0
\]

Exact solution in cylindrical geometry (2D “star”):

\[
\psi \propto b \propto J_m(kr) \exp[im(\phi - \Omega_p t)] \quad k = N_1/\Omega_p
\]

Wave overturns if \( \frac{u_\phi}{r} > \Omega_p \)
BREAKING GRAVITY WAVES

Barker & Ogilvie (2010)

typical wavelength 0.001-0.01 \( R_{\text{Sun}} \)
Stability analysis of gravity waves

Barker & Ogilvie (2011)

Primary wave is a steady non-axisymmetric flow in a rotating frame
Stability analysis of gravity waves

Contains convectively unstable regions if $A > 1$

$A = 1.1$

$\log_{10} \text{Re}[\sqrt{-N^2}]$
Stability analysis of gravity waves

Contains convectively unstable regions if $A > 1$

$A = 10$

$\log_{10} \text{Re}[\sqrt{-N^2}]$
Stability analysis of gravity waves

Stability analysis by spectral (Galerkin) method

\[ \psi = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} \psi_{mn} J_m(k_{kn} r) e^{im\phi - i\omega t} \]

\[ b = \sum_{m=-\infty}^{\infty} \sum_{n=0}^{\infty} b_{mn} J_m(k_{kn} r) e^{im\phi - i\omega t} \]

Include hyperdiffusion to smooth smallest scales
Stability analysis of gravity waves

Results for $A > 1$ : initial stages of wave breaking

Growth rate vs primary amplitude

Barker & Ogilvie (2011)
Stability analysis of gravity waves

Results for $A > 1$: initial stages of wave breaking

Unstable mode for $A = 10$

Barker & Ogilvie (2011)
Stability analysis of gravity waves

Results for $A > 1$ : initial stages of wave breaking

Unstable mode for $A = 10$

Barker & Ogilvie (2011)
Stability analysis of gravity waves

Results for $A > 1$ : initial stages of wave breaking

Unstable mode for $A = 10$

$$\log_{10} \text{Re}[\sqrt{-N^2}]$$
Stability analysis of gravity waves

Results for $A > 1$: initial stages of wave breaking

Growth rate independent of size of domain
Stability analysis of gravity waves

Results for $A < 1$: weak parametric instabilities

Growth rate vs primary amplitude

$log_{10}\left\{ \frac{\text{Im} [\omega]}{\omega_p} \right\}$ vs $\log_{10} A$

Barker & Ogilvie (2011)
Stability analysis of gravity waves

Results for $A < 1$: weak parametric instabilities

Growth rate $\propto (\text{size of domain})^{-1}$
2D numerical simulations
Barker & Ogilvie 2010

Standing wave
2D numerical simulations
Barker & Ogilvie 2010

Breaking wave
3D numerical simulations

Barker & Ogilvie 2011

Standing wave

\[ \tilde{u}_\phi \]
3D numerical simulations

Barker & Ogilvie 2011

Standing wave
3D numerical simulations

Barker & Ogilvie 2011

Breaking wave
3D numerical simulations

Barker & Ogilvie 2011

Breaking wave

\( \tilde{u}_\phi \)
3D numerical simulations

Barker & Ogilvie 2011

Breaking wave

\[ \frac{\Omega}{\Omega_p} \]
3D numerical simulations

Barker & Ogilvie 2011

Breaking wave
Implications

- Waves break at centre if

\[ \frac{M_p}{M_J} > 3.6 \left( \frac{P_{\text{orb}}}{\text{day}} \right)^{-1/6} \]

or more easily in older or slightly more massive stars

- If this occurs, then \( Q_*' \approx 9 \times 10^4 \left( \frac{P_{\text{orb}}}{\text{day}} \right)^{14/5} \)

and planet is devoured within \( 1.4 \text{ Myr} \left( \frac{M_p}{M_J} \right)^{-1} \left( \frac{P_{\text{orb}}}{\text{day}} \right)^{7.1} \)

- For solar-type binary stars, eccentricity tides are likely to break for any observable eccentricity

For smaller forcing amplitudes, resonant locking (Savonije & Witte) may need to be reexamined allowing for wave breaking
Short-period extrasolar planets

![Graph showing the relationship between orbital period and mass for extrasolar planets.](exoplanets.org | 11/30/2011)

- **Msin(i) [Jupiter Mass]**
- **Orbital Period [Days]**

The graph plots the mass of the planets (Msin(i)) against their orbital periods, with red dots representing individual data points. The x-axis represents the orbital period in days, ranging from 0 to 10, while the y-axis shows the mass in Jupiter masses, ranging from 0.001 to 10.
Effective viscosity of turbulent convection and other flows
Tides in convective regions of planets and stars

- Turbulent viscosity acting on tidal bulge?
- Excitation and dissipation of inertial waves?
Effective viscosity of turbulent convection

- How does a convecting fluid respond to periodic distortion?
- Oscillatory shearing box

\[ a \sin(\omega t) \]
Effective viscosity of turbulent convection

- Compute convective or other flow in OSB
- Measure Reynolds stress at frequency $\omega$
Previous hypotheses and results

- Zahn (1966): viscosity $\propto \omega^{-1}$ (large eddies)
- Goldreich & Nicholson (1977): viscosity $\propto \omega^{-2}$ (small eddies)
- Goodman & Oh (1997): viscosity $\propto \omega^{-5/3}$ (small eddies)
- Penev et al. (2009): viscosity $\propto \omega^{-1}$
Convection in an oscillatory shearing box (Geoffroy Lesur)
Time series of Reynolds stress (shear stress)
Convection in an oscillatory shearing box (Geoffroy Lesur)

- Fourier transform of Reynolds stress (shear stress)
Real part of effective viscosity versus tidal frequency

\[ \Re(\nu_\text{e}) \]

\[ \propto \omega^{-2} \]

uncertainty due to noise

(dashed: negative)
Convection in an oscillatory shearing box (Geoffroy Lesur)

- Imaginary part of effective viscosity versus tidal frequency
Analytical approach

- **Shearing coordinates**
  \[ x' = x, \quad y' = y - a(t)x, \quad z' = z, \quad t' = t \]

- **Shear** \( a(t) \), **shear rate** \( \dot{a}(t) \)

- **Derivatives**
  \[ \partial_x = \partial'_x - a \partial'_y, \quad \partial_y = \partial'_y, \quad \partial_z = \partial'_z, \quad \partial_t = \partial'_t - \dot{a}x \partial'_y \]

- **Absolute and relative velocities**
  \[ u_x = v_x, \quad u_y = v_y + \dot{a}x, \quad u_z = v_z \]

- **Navier–Stokes equations in sheared coordinates**
  \[
  [\partial'_t + v_j (\partial'_j - a \delta_{j1} \partial'_y)]v_i + \dot{a} v_x \delta_{i2} = - (\partial'_i - a \delta_{i1} \partial'_y) p \\
  + \nu (\partial'_j - a \delta_{j1} \partial'_y) (\partial'_j - a \delta_{j1} \partial'_y) v_i + f_i \\
  (\partial'_i - a \delta_{i1} \partial'_y) v_i = 0
  \]

- **Additional body force** \( \ddot{a} x' \delta_{i2} \) is required to maintain shear

- **Remaining equations are spatially homogeneous**
Analytical approach

● Linearize in the shear amplitude, assuming $|a| \ll 1$
● Zeroth order: basic flow satisfying

$$\left( \partial'_t + v_j \partial'_j \right) v_i = -\partial'_i p + \nu \Delta' v_i + f_i$$
$$\partial'_i v_i = 0$$

(may be decaying or (quasi-)stationary, laminar or turbulent)

● First order:

$$\left( \partial'_t + v_j \partial'_j \right) \delta v_i + (\delta v_j \partial'_j - a v_x \partial'_y) v_i + \dot{a} v_x \delta_i 2$$
$$= -\partial'_i \delta p + a \delta_i 1 \partial'_y p + \nu (\Delta' \delta v_i - 2a \partial'_x \partial'_y v_i)$$

$$\partial'_i \delta v_i - a \partial'_y v_x = 0$$

● Aim to calculate linearized shear stress $-\delta R_{xy} = -\langle v_x \delta v_y + v_y \delta v_x \rangle$

● Could solve numerically (but not useful for chaotic flows)

● Asymptotic approach for high-frequency shear
Analytical approach

- Asymptotic approach for high-frequency shear
- Method of multiple scales
- Fast time variable for shear: $T' = t'/\varepsilon, \varepsilon \ll 1$
  $a \mapsto a(T'), \dot{a} \mapsto \varepsilon^{-1} \dot{a}$ ( now means $\frac{da}{dT'}$ )
- Expand
  $$\begin{align*}
  \delta v_i &= \delta v_{i0} + \varepsilon \delta v_{i1} + \cdots \\
  \delta p &= \varepsilon^{-1} (\delta p_0 + \varepsilon \delta p_1 + \cdots )
  \end{align*}$$
- Leading order
  $$\begin{align*}
  \partial'_T \delta v_{i0} + \dot{a} v_x \delta_{i2} &= -\partial'_i \delta p_0 \\
  \partial'_i \delta v_{i0} - a \partial'_y v_x &= 0
  \end{align*}$$
- Rough argument:
  $$\begin{align*}
  \delta v_y0 &\approx -a v_x \Rightarrow -\langle v_x \delta v_{y0} \rangle \approx a \langle v_x^2 \rangle \quad \text{elastic stress}
  \end{align*}$$
Analytical approach

- Leading order
  \[ \partial'_T \delta v_{i0} + \dot{\alpha} \nu_x \delta_{i2} = -\partial'_i \delta p_0 \]
  \[ \partial'_i \delta v_{i0} - a \partial'_y \nu_x = 0 \]

- More precise argument
  \[ \Delta' \delta p_0 = -2\dot{\alpha} \partial'_y \nu_x \]
  \[ \partial'_T \delta v_{i0} = 2\dot{\alpha} \partial'_i \partial'_y \Delta'^{-1} \nu_x - \dot{\alpha} \nu_x \delta_{i2} \]

- Linearized shear stress
  \[ -\delta R_{xy0} = -\langle \nu_x \delta \nu_{y0} + \nu_y \delta \nu_{x0} \rangle \]
  satisfies
  \[ \partial'_T (-\delta R_{xy0}) = \dot{\alpha} \langle \nu_x^2 - 2(\nu_x \partial'_y + \nu_y \partial'_x) \partial'_y \Delta'^{-1} \nu_x \rangle \]
  \[ = \dot{\alpha} (A_{1jj1} - 2A_{1221} - 2A_{2121}) \]
  in terms of the tensor
  \[ A_{ijkl} = \langle \nu_i \partial'_j \partial'_k \Delta'^{-1} \nu_l \rangle \]
Analytical approach

- Next order can be treated in a similar way

\[
\partial^2_T (-\delta R_{xy}) = -a(B_{1jj1} - B_{1221} - B_{1122} - C_{1221} + C_{1jj1} + 3C_{1122}

- 2D_{1jj221} - 2D_{2jj121} - 3D_{1jj221} - 3D_{1jj212}

- D_{iijj1221} - D_{iijj1212} + D_{iijj22} + 4E_{2121} + 4E_{2112})
\]

in terms of the tensors

\[
B_{ijkl} = \langle (\partial'_i v_i) \partial'_j \partial'_k \Delta'^{-1} v_l \rangle
\]

\[
C_{ijkl} = -\nu \langle v_i \partial'_j \partial'_k v_l \rangle
\]

\[
D_{ijkl} = \langle v_i v_j \partial'_k v_l \rangle
\]

\[
D_{ijklmn} = \langle v_i v_j \partial'_k \partial'_l \partial'_m \Delta'^{-1} v_n \rangle
\]

\[
D_{ijklmnpq} = \langle v_i v_j \partial'_k \partial'_l \partial'_m \partial'_n \partial'_p \Delta'^{-2} v_q \rangle
\]

\[
E_{ijkl} = \langle v_m (\partial'_m \partial'_n \Delta'^{-1} \partial'_i v_j) \partial'_n \Delta'^{-1} \partial'_k v_l \rangle
\]
Analytical approach

- Interpretation
  \[ \partial'_T (-\delta R_{xy0}) = \dot{a}G_0 \]
  \[ \partial''_T (-\delta R_{xy1}) = -\ddot{a}G_1 \]

- For a shear \( a \propto \exp(-i\omega t) \) with \( \omega = O(\epsilon^{-1}) \), deduce that
  \[ -\delta R_{xy} = a \left[ G_0 - \frac{iG_1}{\omega} + O(\epsilon^2) \right] \]

- Compare with elastic stress \( G_0 \dot{a} \) or viscous stress \( \nu \ddot{a} = -i\omega \nu a \)

- Effective elastic (shear) modulus \( G_0 \) (+, − or 0)
- Effective viscosity at high frequencies \( G_1/\omega^2 \) (+, − or 0)
Analytical approach

- Evaluation in special cases
  - Statistically isotropic flows in $d$ dimensions

\[
G_0 = \frac{2(d - 2)(d + 1)}{d(d - 1)(d + 2)} K \\
G_1 = \frac{(d^2 - 2)\dot{K} + (d^2 - 6)D}{d(d - 1)(d + 2)} \\
K = \langle \frac{1}{2} v_i v_i \rangle \\
\dot{K} = \langle (\partial^t_i v_i) v_i \rangle \\
D = -\nu \langle v_i \Delta^t v_i \rangle
\]

- Thus effective elasticity $> 0$ in 3D but $= 0$ in 2D
- Effective viscosity $> 0$ in 3D but $< 0$ in 2D if flow maintained
  but $< 0$ in 3D or 2D if flow decays freely
Analytical approach

- Evaluation in special cases
  - ABC flows (Arnol’d–Beltrami–Childress)
    \[ \mathbf{v} = \begin{pmatrix} A \sin k z' + C \cos k y' \\ B \sin k x' + A \cos k z' \\ C \sin k y' + B \cos k x' \end{pmatrix} \]
    in a period cube of length \( 2\pi/k \)
  - Nonlinearity absent because of Beltrami property \( \nabla' \times \mathbf{v} = k \mathbf{v} \)
  - If unforced, \( A, B, C \propto \exp(-\nu k^2 t) \)
  - Or maintain flow with body force \( \mathbf{f} = \nu k^2 \mathbf{v} \)
  - Find
    \[ G_0 = \frac{1}{2} (A^2 - C^2) \] (depends on anisotropy)
    \[ G_1 = \frac{1}{2} A (\dot{A} + \nu k^2 A) \] (vanishes if freely decaying)
  - These analytical examples lack genuine nonlinearity / irreversibility
Analytical results for high-frequency shear

General flow (laminar, turbulent, convective, ... )

Tidal period $\ll$ flow timescales (relevant for large eddies)

- Dominant response is elastic
- Next effect is viscosity $\propto \omega^{-2}$
- Coefficients may be positive, negative or zero depending on flow statistics, anisotropy, etc.
- Incompatible with Zahn (1966)
- Different from Penev et al. (2009)
- Raises the possibility of tidal anti-dissipation
Conclusions
Conclusions

- Tidal evolution probably determines the fate of short-period extrasolar planets
- Idealized linear inertial waves give an intricate frequency dependence of $Q'$, still only partly understood
- Frequency-averaged dissipation is robust and readily calculated
- For $l = m = 2$ dissipation is most efficient for:
  - larger, more rigid or denser cores
  - greater density stratification (larger polytropic index)
- Other (tesseral) harmonics excite richer response and may be important even though intrinsically weaker
- Better models of planetary (and stellar) interiors are needed and more understanding of the interaction of tides with convection, magnetic fields, etc.
Conclusions

- Nonlinear aspects (wave breaking, mode coupling, etc.) can be important even for “weak” tides. Extrasolar planets may be in a different regime from solar-system planets.

- Wave breaking can lead to the destruction of sufficiently massive planets orbiting close to solar-type stars at a critical age.

- Effective viscosity of convection is strongly suppressed at tidal frequencies higher than the convective turnover rate.

- Thermal and magnetic tides also require further investigation as well as waves in extrasolar planetary atmospheres.

- Extrasolar systems are diverse and can reveal much when examined on an individual basis.
References


References


References

- Gostiaux, L. and Dauxois, T., 2007: Laboratory experiments on the generation of internal tidal beams over steep slopes, Phys. Fluids 19, 028102.
References


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References


References


References


References


References


