Lecture 2
Quasi-geostrophic waves and transport

(i) Quasigeostrophic equations and potential vorticity
(ii) Wave activity conservation
(iii) Stability of zonal flows
(iv) PV transport and nonacceleration
(v) Mean momentum and heat budgets
(vi) Rossby waves: barotropic, baroclinic, and breaking
(i) Quasigeostrophic equations and potential vorticity
Hydrostatic equations with rotation (log-p coordinates, \( f = 2\Omega \sin \phi \)):

\[
\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u - fv = -\frac{\partial \phi}{\partial x} + G^{(x)}
\]

\[
\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + fu = -\frac{\partial \phi}{\partial y} + G^{(y)}
\]

\[
\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta = (\rho \Pi)^{-1} J
\]

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w) = 0
\]

\[
\frac{\partial \phi}{\partial z} - \frac{\kappa \Pi}{H} \theta = 0
\]

Assumptions:

- Midlatitude “beta-plane” \( f = f_0 + \beta y \)
Hydrostatic equations with rotation (log-p coordinates, \( f = 2\Omega \sin \phi \)):

\[
\begin{align*}
\frac{\partial u}{\partial t} + \mathbf{u} \cdot \nabla u - fv &= -\frac{\partial \phi}{\partial x} + G^{(x)} \\
\frac{\partial v}{\partial t} + \mathbf{u} \cdot \nabla v + fu &= -\frac{\partial \phi}{\partial y} + G^{(y)} \\
\frac{\partial \theta}{\partial t} + \mathbf{u} \cdot \nabla \theta &= (\rho \Pi)^{-1} J
\end{align*}
\]

\[\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w) &= 0 \\
\frac{\partial \phi}{\partial z} - \frac{\kappa \Pi}{H} \theta &= 0
\end{align*}\]

Assumptions:
- Midlatitude “beta-plane” \( f = f_0 + \beta y \)
- \( Ro = UlfL \ll 1 \rightarrow \text{geostrophic balance} \)

\( G \) represents the geostrophic balance term.
Hydrostatic equations with rotation (log-\(p\) coordinates, \(f = 2\Omega \sin \phi\)):

\[
\frac{\partial u}{\partial t} + u \cdot \nabla u - fv = -\frac{\partial \phi}{\partial x} + G^{(x)}
\]
\[
\frac{\partial v}{\partial t} + u \cdot \nabla v + fu = -\frac{\partial \phi}{\partial y} + G^{(y)}
\]
\[
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta = (\rho \Pi)^{-1} J
\]
\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w) = 0
\]
\[
\frac{\partial \phi}{\partial z} - \frac{\kappa \Pi}{H} \theta = 0
\]

Assumptions:

- Midlatitude “beta-plane” \( f = f_0 + \beta y \)
- \( Ro = U/lfL \ll 1 \rightarrow \) geostrophic balance
- \( \beta L/f_0 \ll 1 \rightarrow \) geostrophic flow nondivergent \( \rightarrow w \approx 0 \)
- At leading order \( \partial \theta / \partial z \) is function of \( z \) only (for consistent entropy budget)
Hydrostatic equations with rotation (log-p coordinates, $f = 2\Omega \sin \phi$):

\[
\begin{align*}
\frac{\partial u}{\partial t} + u \cdot \nabla u - fv &= -\frac{\partial \phi}{\partial x} + G^{(x)} \\
\frac{\partial v}{\partial t} + u \cdot \nabla v + fu &= -\frac{\partial \phi}{\partial y} + G^{(y)} \\
\frac{\partial \theta}{\partial t} + u \cdot \nabla \theta &= (\rho \Pi)^{-1} J \\
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w) &= 0 \\
\frac{\partial \phi}{\partial z} - \frac{\kappa \Pi}{H} \theta &= 0
\end{align*}
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Assumptions:

- Midlatitude “beta-plane” $f = f_0 + \beta y$
- $Ro = Ul/fL \ll 1 \rightarrow$ geostrophic balance
- $\beta L / f_0 \ll 1 \rightarrow$ geostrophic flow nondivergent $\rightarrow$ $w \approx 0$
- At leading order $\partial \theta / \partial z$ is function of $z$ only (for consistent entropy budget)
Define background state

\[ \Theta_0(z), \quad \Phi_0(z) = \frac{\kappa}{H} \int_0^z \Pi \Theta_0 \, dz \]

Geostrophic flow:

\[-fv_g = -\frac{\partial \phi}{\partial x}; \quad fu = -\frac{\partial \phi}{\partial y} \]
\[ \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0 \]
\[ u_g = -\frac{\partial \psi}{\partial y}; \quad v_g = \frac{\partial \psi}{\partial x}; \quad w_g = 0 \]

geostrophic streamfunction:

\[ \psi = [\phi - \Phi_0(z)]/f_0 \]

Hydrostatic balance

\[ \frac{\partial \psi}{\partial z} = \frac{\kappa \Pi}{f_0 H} [\theta - \Theta_0(z)] \]

\[ \to \] thermal wind shear

\[ f_0 \frac{\partial u}{\partial z} = -\frac{\kappa \Pi}{f_0 H} \frac{\partial \theta}{\partial y}; \quad f_0 \frac{\partial v}{\partial z} = \frac{\kappa \Pi}{f_0 H} \frac{\partial \theta}{\partial x} \]
Quasi-geostrophic equations 2

At next order,

\[ D_g u_g - \beta y v_g - f_0 v_a = G^{(x)} \]  
(1)
\[ D_g v_g + \beta y u_g + f_0 u_a = G^{(y)} \]  
(2)
\[ D_g \theta + w_a \frac{\partial \Theta_0}{\partial z} = (\rho \Pi)^{-1} J \]  
(3)

\[ \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho w_a) = 0 \]

where \( D_g \) is derivative following geostrophic flow:

\[ D_g \equiv \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y} \]

and \((u_a, v_a, w_a)\) is the ageostrophic velocity

\((u_a, v_a, w_a) = (u - u_g, v - v_g, w)\)
From these, we can derive
\[ \{ \partial(2)/\partial x - \partial(1)/\partial y + (f_0/\rho)\partial(\rho \times [3]/\Theta_{0,z})/\partial z \} \]
the equation for quasigeostrophic potential vorticity, \( q \):

\[ \rightarrow \quad D_g q = X \]

where

\[
q = f_0 + \beta y + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} + \frac{f_0}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{\partial}{\Theta_{0,z}} \right)
\]

\[
= f_0 + \beta y + \left[ \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{f_0^2}{N^2} \frac{\partial}{\partial z} \right) \right] \psi
\]

and

\[
X = \frac{\partial G^{(y)}}{\partial x} - \frac{\partial G^{(x)}}{\partial y} + \frac{f_0}{\rho} \frac{\partial}{\partial z} \left( \rho \frac{J}{\Pi \Theta_{0,z}} \right)
\]

\[ \rightarrow \text{ for conservative flow (} \mathbf{G} = 0, J = 0, \text{ whence } X = 0\) : \( q \) is conserved following the geostrophic flow.
(ii) Wave activity conservation
PV fluxes and the Eliassen-Palm theorem
Consider small-amplitude motions on a steady, zonally-uniform basic state

\[ [u_g, v_g, w] = [U(y, z), 0, 0] ; \theta = \Theta(y, z) ; \psi = \Psi(y, z) ; Q(y, z) \]

where

\[ \frac{\partial \Psi}{\partial y} = -U ; \quad \frac{\kappa \Pi H}{\partial y} = -f_0 \frac{\partial U}{\partial z} \]

\[ Q(y, z) = f_0 + \beta y + \frac{\partial^2 \Psi}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\rho f_0^2}{N^2} \frac{\partial \Psi}{\partial z} \right) \]

Write

\[ \psi = \Psi + \psi'(x, y, z, t) \]

then \( v' = \partial \psi'/\partial x \) and

\[ q' = \Delta^2 \psi' = \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \right) . \]

so PV flux is

\[ \overline{v'q'} = \frac{\partial \psi'}{\partial x} \left[ \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \right) \right] \]
Consider $v' q'$:

(I) $\frac{\partial \psi'}{\partial x} \frac{\partial^2 \psi'}{\partial x^2} = \frac{1}{2} \frac{\partial}{\partial x} \left[ \left( \frac{\partial \psi'}{\partial x} \right)^2 \right] = 0$;

(II) $\frac{\partial \psi'}{\partial x} \frac{\partial^2 \psi'}{\partial y^2} = \frac{\partial}{\partial y} \left[ \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial y} \right] - \frac{\partial \psi'}{\partial y} \frac{\partial^2 \psi'}{\partial x \partial y}$

$= \frac{\partial}{\partial y} \left[ \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial y} \right] - \frac{1}{2} \frac{\partial}{\partial x} \left[ \left( \frac{\partial \psi'}{\partial y} \right)^2 \right]$;

(III) $\frac{\partial \psi'}{\partial x} \frac{1}{\rho} \frac{\partial}{\partial z} \left[ \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \right] = \frac{1}{\rho} \frac{\partial}{\partial z} \left[ \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial z} \right] - \frac{f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \frac{\partial^2 \psi'}{\partial x \partial z}$

$= \frac{1}{\rho} \frac{\partial}{\partial z} \left[ \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial z} \right] - \frac{f_0^2}{2N^2} \frac{\partial}{\partial z} \left[ \left( \frac{\partial \psi'}{\partial z} \right)^2 \right]$;

$= \frac{1}{\rho} \frac{\partial}{\partial z} \left[ \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial z} \right]$. 
Therefore

\[ \rho v'q' = \nabla \cdot \mathbf{F} \]

where

\[ \mathbf{F} = (F^{(y)}, F^{(z)}) \]

\[ = \left( \rho \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial y}, \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial z} \right) \]

\[ = \left( -\rho u'v', \rho f_0 \frac{v'\theta'}{d\Theta_0/dz} \right) \]

\( \mathbf{F} \) is known as the ELIASSEN-PALM flux.
Linearizing the QGPV equation:

\[
\left(\frac{\partial}{\partial t} + U \frac{\partial}{\partial x}\right) q' + v' \frac{\partial Q}{\partial y} = X'
\]
multiply by \( q' \) and average:

\[
\frac{\partial}{\partial t} \left(\frac{1}{2} q'^2\right) + \overline{v' q'} \frac{\partial Q}{\partial y} = \overline{v' X'}
\]

Define

\[
A = \rho \frac{1}{2} \overline{q'^2} \left(\frac{\partial Q}{\partial y}\right) \text{ and } \quad D = \rho \overline{v' X'} \left(\frac{\partial Q}{\partial y}\right),
\]

\[
\frac{\partial A}{\partial t} + \nabla \cdot \mathbf{F} = D
\]

\( \rightarrow \) the ELIassen-Palm relation:
— a conservation law for zonally-averaged wave activity
whose density is \( A \). Note that \( D \rightarrow 0 \) for conservative flow.

\( \mathbf{F} \) is a meaningful measure of the propagation of wave activity

**The Eliassen-Palm theorem**

For steady \( (\partial A/\partial t = 0) \), small amplitude, conservative \( (D = 0) \) waves:

\[
\nabla \cdot \mathbf{F} = 0 \quad : \quad \rho \overline{v' q'} = 0
\]
(iii) Stability of zonal flows
Stability of zonal flows to QG perturbations: The Charney-Stern theorem

Integrate the EP relation:

\[
\frac{\partial}{\partial t} \int \int_{\mathcal{R}} A \, dy \, dz + \oint_{\mathcal{C}} \mathbf{F} \cdot \mathbf{n} \, dl = \int \int_{\mathcal{R}} D \, dy \, dz
\]

over the domain \( \mathcal{R} \) bounded by the surface.

Boundary fluxes:
- at sides \( y = y_1, y_2, \quad v = 0 \):
  \[\rightarrow \quad \mathbf{F} \cdot \mathbf{n} = F^{(y)} = -\rho u'v' = 0\]
- at top and bottom:
  \[\mathbf{F} \cdot \mathbf{n} = F^{(z)} = \rho f_0 \frac{\overline{v'\theta'}}{d\Theta_0/dz}\]

which is nonzero if \( \overline{v'\theta'} \neq 0 \). But if the upper and lower boundaries are isentropic, then

\[\theta' = 0 \rightarrow \mathbf{F} \cdot \mathbf{n} = 0\]

there.
Hence for
(i) conservative flow (no creation or dissipation of wave activity)
(ii) with isentropic upper and lower boundaries
(no flux through boundaries)

\[ \frac{\partial}{\partial t} \iint_{\mathcal{R}} A \, dy \, dz = 0 \]

→ globally integrated wave activity is conserved.
But sign of \( A \) depends on sign of \( \partial q/\partial y \):

\[ A = \frac{1}{2} \rho \overline{q^{1/2}} \]

Look for normal mode growth such that \( \overline{q^{1/2}} = B(t)C(y, z) \)
(both \( B \) and \( C \) positive definite)

\[ \frac{dB}{dt} \iint_{\mathcal{R}} \frac{1}{2} \frac{C(y, z)}{\partial q/\partial y} \, dy \, dz = 0 \]

If mean PV gradient is single-signed, \( dB/dt = 0 \) → no growth

Hence
A zonal flow is stable to inviscid, adiabatic, quasigeostrophic normal mode perturbations if

- **a.** there is no change of sign of PV gradient within the fluid and
- **b.** the system is bounded above and below by isentropic boundaries.

The *Charney-Stern theorem*. (does not apply to non-normal-mode growth).
(iv) PV transport and nonacceleration
Potential vorticity transport and the nonacceleration theorem

How do eddies influence the zonal mean circulation?
Take mean of QGPV equation

$$\frac{\partial \bar{q}}{\partial t} + \frac{\partial}{\partial y} (\bar{v} q') = \bar{X}.$$ 

Note (i) \(v_g = \frac{\partial \psi}{\partial x} = 0\), so no mean advection
(ii) \(w_g = 0\), so no vertical eddy flux to leading order
→influence of eddies described entirely by the northward flux \(v' q' = \rho^{-1} \nabla \cdot \mathbf{F}\)

Know from the Eliassen-Palm theorem that if the waves are everywhere
(I) of small amplitude,
(II) conservative, and
(III) statistically steady
→\(\mathbf{F}\) is nondivergent and \(v' q' = 0\). Then \(\partial q/\partial t\) is independent of the waves
(if we assume that \(\bar{X}\) is also independent).
Then $\partial \tilde{q}/\partial t$ is independent of the waves (if we assume that $\tilde{X}$ is also independent). Now,

$$\tilde{q} = f + \Delta^2(\psi)$$

therefore can invert PV:

$$\frac{\partial \psi}{\partial t} = \Delta^{-2} \frac{\partial \tilde{q}}{\partial t} = \Delta^{-2} \tilde{X}$$

$\Delta^2$ is an elliptic operator, so solution invokes boundary conditions on $\partial \psi/\partial t$. If we invoke the further condition that

(IV) the boundary conditions on $\partial \psi/\partial t$ are independent of the waves then $\partial \psi/\partial t$ is everywhere independent of the waves.

$\tilde{u} = -\partial \psi/\partial y$, $\tilde{\theta} = (f_0 H/\kappa \Pi) \partial \psi/\partial z$ → same true of $\partial \tilde{u}/\partial t$, $\partial \tilde{\theta}/\partial t$. 

→ nonacceleration theorem (Charney-Drazin, Andrews-McIntyre)

Closely related to Kelvin’s circulation theorem:
(v) Mean momentum and heat budgets
Mean momentum and heat budgets

Zonal mean QG eqs:

\[
\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}_a = G^{(x)} - \frac{\partial}{\partial y} (u'v')
\]

\[
\frac{\partial \bar{\theta}}{\partial t} + \bar{w}_a \frac{\partial \bar{\theta}}{\partial z} = (\rho \Pi)J - \frac{\partial}{\partial y} (\bar{v}'\bar{\theta}')
\]

\[
\frac{\partial \bar{v}_a}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho \bar{w}_a) = 0
\]

\[
f_0 \frac{\partial \bar{u}}{\partial z} + \frac{\kappa \Pi}{f_0 H} \frac{\partial \bar{\theta}}{\partial y} = 0
\]

set of 4 equations in the 4 unknowns $\partial \Pi/\partial t$, $\partial T/\partial t$, $\bar{v}_a$ and $\bar{w}_a$
in terms of the two eddy driving terms $u'v'$, $v'\theta'$

Central role of the PV flux—obvious in mean PV budget—not obvious here
**Transformed Eulerian-mean theory**  

Define ageostrophic “residual” mean streamfunction

\[
(\bar{v}_*, \bar{w}_*) = \left[ \bar{v}_a - \frac{1}{\rho} \frac{\partial (\rho \chi_*)}{\partial z}, \bar{w}_a + \frac{\partial \chi_*}{\partial y} \right]
\]

where

\[
\chi_* = \frac{\bar{v}' \bar{\theta}'}{\partial \theta / \partial z}
\]

(and remember \(\bar{\theta} = \bar{\theta}(z)\) to leading order). Then

\[
\frac{\partial \bar{u}}{\partial t} - f_0 \bar{v}_a = G^{(x)} - \frac{\partial}{\partial y} (u' v') \\
\frac{\partial \bar{\theta}}{\partial t} + w_a \frac{\partial \bar{\theta}}{\partial z} = (\rho \Pi) J - \frac{\partial}{\partial y} (v' \theta') \\
\frac{\partial \bar{v}_a}{\partial y} + \frac{1}{\rho} \frac{\partial}{\partial z} (\rho \bar{w}_a) = 0 \\
f_0 \frac{\partial \bar{u}}{\partial z} + \frac{\kappa \Pi}{f_0 H} \frac{\partial \bar{\theta}}{\partial y} = 0
\]

where \(\bar{u} = \nabla \times \mathbf{F}\) is the EP flux, as before.  
Now have set of equations for \(\bar{v}_*, \bar{w}_*, \partial \bar{u}/\partial t\) and \(\partial \bar{T}/\partial t\) in terms of one eddy forcing term \(\rho^{-1} \nabla \cdot \mathbf{F} = v' q'\), appearing as effective body force (per unit mass).  
Nonacceleration theorem then follows directly.
\[ F = (F^{(y)}, F^{(z)}) \]
\[
= \left( \rho \frac{\partial \psi'}{\partial x} \frac{\partial \psi'}{\partial y}, \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial y} \frac{\partial \psi'}{\partial z} \right) \\
= \left( -\rho u'v', \rho f_0 \frac{v' \theta'}{d \Theta_0/dz} \right)
\]
**F** as a momentum flux:

Consider adiabatic flow; isentropic surface \( C \) (of constant \( \theta \)), disturbed by small-amplitude waves. Zonally-averaged zonal stress on \( C \) is \( \tau \) where

\[
\rho \tau = -\bar{p} \sin \gamma \approx -p \gamma \approx -\gamma \frac{\partial}{\partial z} \delta p
\]

(since \( \bar{\gamma} = 0 \)) where \( \delta p \) is the pressure variation along \( C \).

\( \gamma \) is small

\[
\rightarrow \tan \gamma \approx \gamma \approx \frac{\partial (\delta z_g)}{\partial x} \approx -\frac{\partial \theta}{\partial x} \frac{\partial \theta}{\partial z}
\]

so \( \delta z_g \approx -\theta'/(\partial \theta/\partial z) \).

\( C \) is the surface of constant geometric height \( z_g \) reference position for \( C \).

\( p' \) the pressure variation along \( C \), then, along \( C \), \( \delta p = p' - g \rho \delta z_g \). So

\[
\frac{\gamma}{\delta p} = \frac{\partial (\delta z_g)}{\partial x} p' - g \rho \frac{\partial (\delta z_g)}{\partial x} \delta z_g = \frac{\partial (\delta z_g)}{\partial x} p' = -\delta z_g \frac{\partial p'}{\partial x} = f_0 \rho \frac{v' \theta'}{\partial \theta/\partial z},
\]

\[
\rightarrow \tau = -f_0 \frac{v' \theta'}{\partial \theta/\partial z}
\]

\( \rightarrow \) so \( F(z) \) represents vertical momentum transport by form drag on isentropic surfaces.

So (unlike *e.g.*, chemical tracers) momentum can be *radiated* over large distances.
(vi) Rossby waves

- Barotropic
- Baroclinic
- Rossby wave breaking
Barotropic Rossby waves

Two-dimensional flow ($\partial/\partial z = 0$)

PV is just absolute vorticity $q = f_0 + \beta y + \nabla^2_h \psi$
($\nabla^2_h = \partial^2/\partial x^2 + \partial^2/\partial y^2$)

Vorticity conservation for waves on a constant zonal flow $\bar{u}$, $
\rightarrow \partial \psi/\partial y = \beta$

$$\frac{\partial q'}{\partial t} + \bar{u} \frac{\partial q'}{\partial x} + \beta v' = 0$$

$$\rightarrow \left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \nabla^2 \psi' + \beta \frac{\partial \psi'}{\partial x} = 0$$

wave solutions $\psi' = \text{Re}[\Psi_0 \exp\{i(kx + ly - k\bar{c}t)\}]$ where

$$c = \bar{u} - \frac{\beta}{k^2 + l^2}$$

“elasticity” of PV gradient
$\rightarrow$ westward propagation (relative to mean flow)
$\rightarrow$ dispersive
Stationary Rossby waves

Long-term January mean geopotential height

Barotropic stationary waves:

\[ c = \bar{u} - \frac{\beta}{k^2 + l^2} \quad \rightarrow \quad \kappa_s^2 = k^2 + l^2 = \frac{\beta}{\bar{u}} \]

For \( \beta = 1.6 \times 10^{-11} \text{m}^{-1} \text{s}^{-1} \), \( \bar{u} = 30 \text{ms}^{-1} \),

\[ \frac{2\pi}{\kappa_s} = 2\pi \sqrt{\frac{\bar{u}}{\beta}} \approx 8600 \text{ km} \]

\( \approx \) zonal wave 3 at 45° latitude

Zonal group velocity of stationary waves:

\[ c_{g,x} = \frac{\partial(ck)}{\partial k} = \bar{u} + \frac{\beta(k^2 - l^2)}{(k^2 + l^2)^2} \]

\[ = 2k^2 \frac{\bar{u}^2}{\beta} > 0 \]
Rossby wave propagation on the sphere from a localized midlatitude source [Held 1983]

Realistic zonal winds (with tropical easterlies)
Stationary Rossby waves in the lab
Critical layers and Rossby wave breaking

\[ \bar{u} \approx \Lambda y \]

\[ \psi = -\frac{1}{2} \Lambda y^2 + \Psi(0) \cos kx \]

Mean westerlies: wavy streamlines

Closed eddies: overturning:

width = \( 4\sqrt{\frac{\Psi(0)}{\Lambda}} \)

\[ \frac{\partial q}{\partial y} > 0 \rightarrow v'q' < 0 \rightarrow \nabla \cdot \mathbf{F} < 0 \]

- absorption of wave activity
Rossby wave propagation on the sphere from a localized midlatitude source [Held 1983]

Realistic zonal winds (with tropical easterlies)
Subtropical breaking of Rossby waves from a localized midlatitude source
(1-layer; 300 hPa mean wind)

[Esler et al., *J Atmos Sci*, 2000]
Subtropical breaking of Rossby waves from a localized midlatitude source
(1-layer; 300 hPa mean wind)

[Esler et al., *J Atmos Sci*, 2000]

PV contours
Baroclinic Rossby waves: Vertical propagation

Conservative, small amplitude waves on constant background flow \( \bar{u} \), \( N^2 \) also constant
Linearized QGPV equation:

\[
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) q' + v' \frac{\partial q}{\partial y} = 0
\]

where now

\[
q' = \Delta^2 \psi' = \frac{\partial^2 \psi'}{\partial x^2} + \frac{\partial^2 \psi'}{\partial y^2} + \frac{1}{\rho} \frac{\partial}{\partial z} \left( \frac{\rho f_0^2}{N^2} \frac{\partial \psi'}{\partial z} \right),
\]

\[
\frac{\partial q}{\partial y} = \beta
\]

so

\[
\left( \frac{\partial}{\partial t} + \bar{u} \frac{\partial}{\partial x} \right) \Delta^2 \psi' + \beta \frac{\partial \psi'}{\partial x} = 0
\]

Recall \( \rho = \rho_0 \exp(-z/H) \). Solutions are of the form

\[
\psi' = \text{Re} \Psi_0 \exp\left(\frac{z}{2H}\right) \exp[i(kx + ly + mz - kct)]
\]

where

\[
m^2 = \frac{N^2}{f_0^2} \left( \frac{\beta}{(\bar{u} - c)^2} - k^2 - l^2 \right) - \frac{1}{4H^2}
\]

or

\[
c - \bar{u} = -\beta \left( k^2 + l^2 + \frac{f_0^2}{N^2} m^2 + \frac{f_0^2}{4N^2H^2} \right)^{-1}
\]

→ dispersion relation for baroclinic Rossby waves
Vertical propagation of stationary waves

Vertical wavenumber $m$ for $c = 0$

$$m^2 = \frac{N^2}{\beta \frac{\beta}{u} - k^2 - l^2} - \frac{1}{4H^2}$$

real $m$ requires $0 < \bar{u} < U_c$

“Rossby critical velocity” $U_c$ is

$$U_c = \beta \left( k^2 + l^2 + \frac{f_0^2}{4N^2H^2} \right)^{-1}$$

$\rightarrow$ propagation “window” for the mean winds
$\rightarrow$ no propagation through easterlies $\bar{u} < 0$, nor strong westerlies $\bar{u} > U_c$

$U_c$ decreases with increasing $k^2 + l^2$, so the window becomes narrow for small-scale waves

synoptic scale wave, $\kappa^2 = 1.96 \times 10^{-11} m^{-2}$, $U_c \approx 1 ms^{-1}$

largest planetary scale wave $k = \pi/(14000 km), l = \pi/(6000 km), U_c \approx 35 ms^{-1}$
Typical stratospheric analyses (30hPa, 2006 Jan 10)

summer

almost no waves

winter

planetary scales only
References