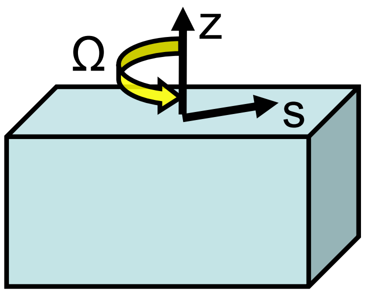
Convection in rotating systems

Consider rotating frame of reference with rotation vector $\mathbf{\Omega}=\Omega\mathbf{e}_z$ (s is coordinate $\perp$ $\mathbf{e}_z$). Two (pseudo-)force terms (Coriolis and centrifugal) are added to Navier-Stokes equation: $D\mathbf{u}_R/Dt = D\mathbf{u}_I/dt + 2\Omega\,\mathbf{e}_z \times \mathbf{u}_R + \Omega^2 s\mathbf{e}_s$. Centrifugal acceleration $\Rightarrow$ gradient of a potential, can be added to gravitational potential $\Rightarrow$ modified gravity vector $\mathbf{g}' = \mathbf{g} - \Omega^2 s\mathbf{e}_s$ (drop prime later).

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + 2\rho\Omega\, e_z \times \mathbf{u} + \nabla p = \eta\,\nabla^2\mathbf{u} + \rho\,\mathbf{g}'$$

Inertia — Coriolis — pressure — viscosity — mod. gravity



To estimate relative size of terms, scale velocity by U, length by D and time by D/U, to obtain non-dimensional equation (gravity omitted):

$$Ro\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + 2\,e_z \times \mathbf{u} + \nabla p = E\,\nabla^2\mathbf{u}$$

$$Ro = \frac{U}{\Omega D} \qquad E = \frac{\nu}{\Omega D^2}$$

Rossby number — Ekman number

Earth's core: $\Omega = 7.6\times10^{-5}\,s^{-1}$, U = 0.5 mm/s, D = 2000 km, $\nu = 2\times10^{-6}\,m^2/s$ $\Rightarrow$ $Ro = 3\times10^{-6}$ $E \approx 10^{-14}$

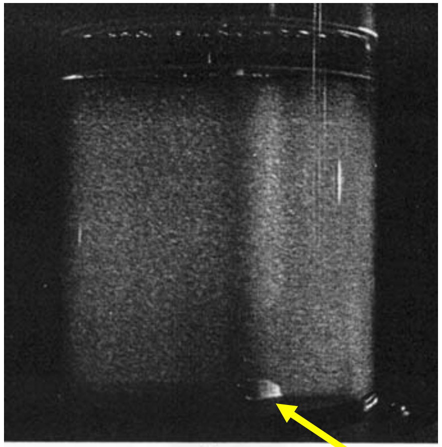
Jupiter's outer shell: $\Omega = 1.7\times10^{-4}\,s^{-1}$, U = 100 m/s, D = 10^4 km, $\nu = 10^{-5}\,m^2/s$ $\Rightarrow$ Ro = 0.05 $E \approx 10^{-15}$

Symbols (index R - rotating reference frame, I – inertial frame): Ω – rotation frequency, g' – modified gravity, U – characteristic velocity, D – characteristic length (e.g. layer height), Ro – Rossby number, E – Ekman number

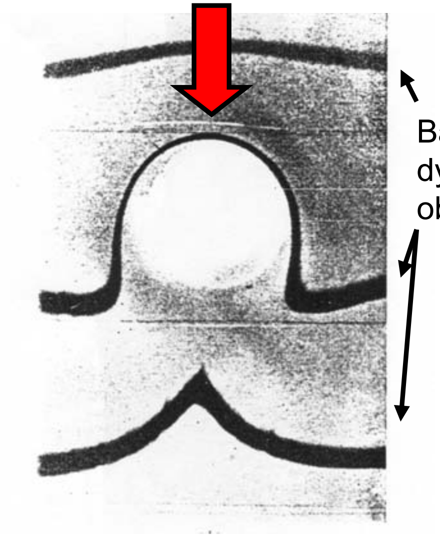
Proudman-Taylor theorem

When both E << 1 and Ro << 1, the dominant force balance is between pressure gradient and Coriolis force (**geostrophic balance**): $2\mathbf{e}_z \times \mathbf{u} = \nabla p$. Take curl, use rule $\nabla\times(\mathbf{a}\times\mathbf{b}) = (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b} + \mathbf{a}\,\nabla\cdot\mathbf{b} - \mathbf{b}\,\nabla\cdot\mathbf{a}$, to obtain for $\nabla\cdot\mathbf{u}=0$

$$\frac{\partial \mathbf{u}}{\partial z} = 0 \qquad \left(likewise\ \frac{\partial p}{\partial z} = 0\right)$$



Proudman-Taylor theorem: flow does not change in the direction of the rotation axis

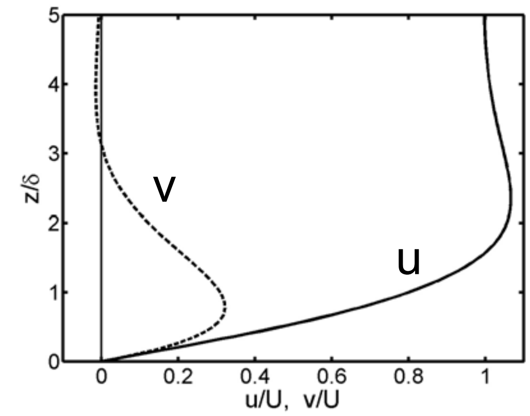


- If the fluid is bounded by impenetrable walls in z-direction $\Rightarrow$ $u_z = 0$ everywhere.
- 2D-streamlines follow isobaric lines (p=const) and $|\mathbf{u}| \sim |\nabla p|$ (geostrophic flow)
- In practice, the Proudman-Taylor theorem is often not satisfied perfectly, but the fluid has a „dynamical stiffness“ along to the rotation axis

Ekman boundary layer

At no-slip boundaries **u** must drop to zero. If they are not parallel to $\mathbf{e}_z$, the P-T-theorem cannot be satisfied near the boundary. In order to violate it, some other force term must be of the same order as the Coriolis/pressure term. Even for a small value of E, the viscous term becomes large if the velocity changes on a small length scale

$\delta \sim E^{1/2}$ in dimensional units: $\delta = D\, E^{1/2} = (\nu/\Omega)^{1/2}$.



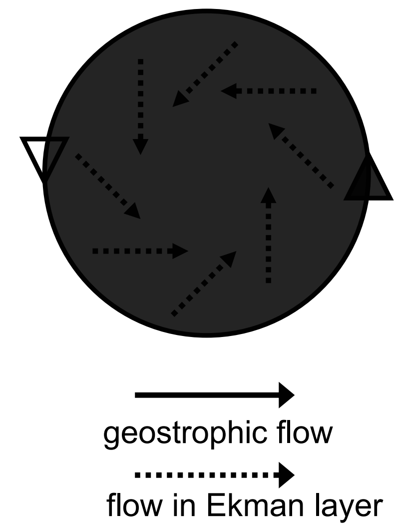
δ is the **Ekman layer thickness**. For a geostrophic flow **u** = (u,v,w) = (U,0,0) far away from the boundary, the solution to the Navier-Stokes equation near the boundary is:

$$u = U\,(1 - \exp(-z/\delta))\cos(z/\delta) \qquad v = U \exp(-z/\delta)\sin(z/\delta)$$

Near the rigid boundary, the flow forms a 45° angle to the geostrophic flow.

When the geostrophic flow forms a vortex that rotates in the same sense as the basic rotation (associated with low pressure), there is a net flow towards the center of the vortex in the Ekman layer (**Ekman suction or Ekman pumping**). There must be a vertical flow away from the boundary.

The opposite occurs for an anticyclonic vortex.



In the Earth's core the Ekman layer thickness is δ ≈ 0.2 m. If the flow becomes turbulent, the boundary can become much thicker.

Symbols: δ – Ekman layer thickness, u,v,w – cartesian velocity components, U – geostrophic flow velocity

Rotating convection: poloidal / toroidal decomposition

Assume that gravity is parallel to rotation axis: **g** = -g e_z. U not known a-priori - take D^2/κ as basic time scale (as before). Assume free-slip boundaries and T(z=0)=1, T(z=1)=0. This time do not make the assumption of 2D-flow.

$$\frac{1}{\Pr}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \frac{2}{E}\, e_z \times \mathbf{u} + \nabla p \;=\; \nabla^2\mathbf{u} + Ra\, T\, \mathbf{e}_z \qquad \nabla\cdot\mathbf{u} = 0$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T$$

Represent velocity by poloidal potential Φ and toroidal potential Ψ (satisfies ∇·u=0 implicitly):

$\mathbf{u} = \nabla\times[\nabla\times\Phi\mathbf{e}_z] + \nabla\times\Psi e_z$ $\quad$ $(u,v,w) = (\partial_{xz}\Phi+\partial_y\Psi,\ \partial_{yz}\Phi-\partial_x\Psi,\ \Delta_2\Phi)$ where $\Delta_2 := \partial_{xx}+\partial_{yy}$

The toroidal part of the flow $\mathbf{u}_{tor}$ has no z-component and $\nabla\times\mathbf{u}_{Pol}$ has no radial component. Linearize the Navier-Stokes equation and take (1) the curl of the equation and (2) the curl of the curl and consider the z-component in both cases:

$$\frac{1}{\Pr}\frac{\partial}{\partial t}\Delta_2\Psi = \nabla^2\Delta_2\Psi + \frac{2}{E}\frac{\partial}{\partial z}\Delta_2\Phi \qquad \frac{1}{\Pr}\frac{\partial}{\partial t}\nabla^2\Delta_2\Phi = \nabla^4\Delta_2\Phi - Ra\,\Delta_2 T - \frac{2}{E}\frac{\partial}{\partial z}\Delta_2\Psi$$

Two scalar equations. Without the Coriolis term (i.e. for E→∞) the two equations decouple.

Symbols: (Notation: $\partial_x := \partial/\partial x$, $\Delta_2 = \partial_{xx}+\partial_{yy}$) Ψ- toroidal vector potential, Φ – poloidal potential

Rotating convection: Linear stability

Set $w = -\Delta_2\Phi$; $\zeta = -\Delta_2\Psi = (\nabla\times u)_z$; $T = (1-z) + \theta$. Linearize temperature equation:

$$\frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\zeta = \nabla^2\zeta + \frac{2}{E}\frac{\partial}{\partial z}w; \qquad \frac{1}{\mathrm{Pr}}\frac{\partial}{\partial t}\nabla^2 w = \nabla^4 w + Ra\,\Delta_2\theta - \frac{2}{E}\frac{\partial\zeta}{\partial z}; \qquad \frac{\partial\theta}{\partial t} = w + \nabla^2\theta$$

Isothermal, impenetrable free-slip boundaries: $w = \partial_{zz}w = \partial_z\zeta = \theta = 0$ at $z=0$, $z=1$.

Expansion in normal modes: $\theta(x,y,z) = \Theta_{klm}\exp(ik_x x+ik_y y+\sigma t)\sin(m\pi z)$

Satisfies boundary conditions. $w(x,y,z) = W_{klm}\exp(ik_x x+ik_y y+\sigma t)\sin(m\pi z)$

$m>0$ integer. k_x, k_y real, σ complex. $\zeta(x,y,z) = Z_{klm}\exp(ik_x x+ik_y y+\sigma t)\cos(m\pi z)$

For $E\rightarrow\infty$ we recover the Rayleigh-Benard case, where $\zeta=0$. At the critical onset of convection, real(σ)=0. General case: $\sigma=i\omega$, consider here only case $\sigma=0$. Set $K^2 = k_x^2+k_y^2$.

$$\frac{E}{2}\frac{K^2+m^2\pi^2}{m\pi}Z = W; \qquad (K^2+m^2\pi^2)\Theta = W; \qquad \frac{2}{E}m\pi\,Z + (K^2+m^2\pi^2)^2 W = Ra\,K^2\,\Theta$$

Replace Z and Θ in 3rd equation by W. Divide by W to obtain critical Rayleigh number:

$$Ra_c(K,m) = \frac{(K^2+m^2\pi^2)^3}{K^2} + \frac{4}{E^2}\frac{m^2\pi^2}{K^2}$$

Symbols: w – vertical velocity, ζ – z-component of vorticity, θ – temperature perturbation, k_x,k_y – wave number components, K=|**k**|, σ – growth rate.

Rotating convection: Linear stability II

Search minimum of $Ra_c(K,m)$ (obviously at m=1): apply $K^4\, d/d(K^2)$ and set to zero:

$$3(K^2+\pi^2)^2 K^2 - (K^2+\pi^2)^3 - 4E^{-2}\pi^2 = 0$$

The critical wavenumber is found as root of 3rd-order equation in K^2. Consider here only asymptotic case E << 1. The 3rd term is very large, so K_{crit} must be large and only the leading power in K needs to be retained, $2K^6 \approx 4E^{-2}\pi^2$. Insert K_{crit} into expression for Ra_c, keeping again only the leading power:

$$Ra_{crit} = 3(2\pi^2)^{2/3} E^{-4/3} \approx 21.9\, E^{-4/3} \qquad K_{crit} = (2\pi^2)^{1/6} E^{-1/3} \approx 1.64 E^{-1/3}$$

1) At low Ekman number, the critical Rayleigh number is very large. The wavenumber becomes large and the aspect ratio of a convection cell, $a=\pi/K_{crit}$, becomes small.

E >> 1	(Benard convection):	$Ra_{crit} = 657.5$	a = 1.414
$E = 10^{-6}$	(lab experiments):	$Ra_{crit} = 2\times10^9$	a = 0.020
$E = 10^{-14}$	(core, gas planets)	$Ra_{crit} = 2\times10^{21}$	$a = 4\times10^{-5}$

Rotational forces strongly inhibit convection. *Why is this so?*
⇒ Because the flow must violate the Proudman-Taylor theorem !

Symbols: a – aspect ratio of convection cell.

Discussion of linear stability results II

2) If Ra_{crit} is so high in the Earth's core, can it convect at all ? Calculate critical ΔT, assuming appropriate values of other properties in the Rayleigh number ($\nu=2\times10^{-6}$, $\kappa=5\times10^{-6}$, $\alpha=10^{-5}$, $g=7$, $D=2.3\times10^{6}$ in SI units):

$\Delta T = 2\times10^{21}\, \kappa\nu/(\alpha g D^3) \approx 3\times10^{-5}\ K^{-1}$ $\Rightarrow$ very small ΔT enough for supercriticality.

3) The condition of criticality can be written as $Ra\, E^{4/3} > 22$, or in terms of physical parameters:

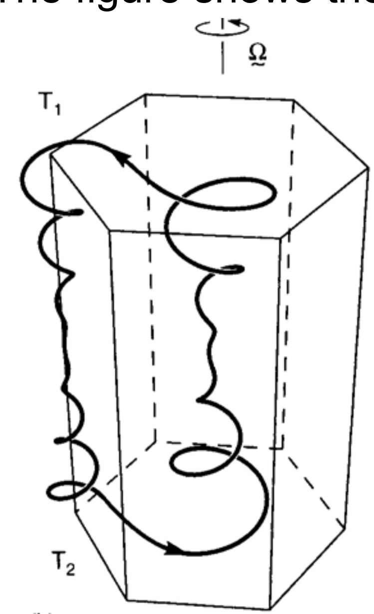
$$\frac{\alpha\, g\, \Delta T\, D^{1/3}\, \nu^{1/3}}{\kappa\, \Omega^{4/3}} > 21.9$$

- unnamed non-dimensional number on left side
- layer thickness D has now only a small influence on the onset of convection
- increasing Ω impedes onset of convection
- increasing ν favors convection (**paradox**). *Why ?* Viscous friction allows for deviation from the Proudman-Taylor theorem. This is also the reason for the small aspect ratio, since the frictional term is $\nu\nabla^2 \mathbf{u} \sim \nu U/a^2$
- an inviscid fluid, $\nu\rightarrow0$, is stable for any temperature contrast

Discussion of linear stability results III

4) As in Benard convection, the 3-D pattern is not determined by the linear stability analysis (different k_x, k_y that result in the same K give the same result). 2D solutions (e.g. $k_y=0$) are possible, but for finite E the velocity is not zero in y-direction.

5) When K is large the ratio between Z and W is $Z = 2\pi/E\, K^{-2}\, W$. The toroidal velocity $\mathbf{u}^T$ relates to ζ by $\zeta=[\nabla\times\mathbf{u}^T]_z$. The amplitude of the toroidal flow V^T in a modal representation is $V^T = Z/K$, therefore $V^T = 2\pi/E\, K^{-3}\, W = 2^{1/2}\, W$ at $K=K_{crit}$. **At low E, toroidal and poloidal velocities have similar amplitude.** The toroidal flow changes sign at mid-depth (z=1/2) and rotates, in rising flow (w>0) in the same sense as the basic rotation for z<1/2 and opposite for z>1/2. The figure shows the path of a fluid particle moving in a hexagonal convection cell.



(b)

6) The product $H = \mathbf{u}\cdot(\nabla\times\mathbf{u})$ is called helicity. The convective flow at low E is very helical, with helicity changing sign at mid-depth. Helicity plays an important role for the dynamo process in a conducting fluid.

7) When we restrict the analysis to real values of σ, the result is independent of the Prandtl number. A more complete analysis shows that indeed Im(σ)=0 at Prandtl number Pr > 0.68. At Pr < 0.68, oscillating solutions are preferred at onset of convection (σ=iω), which can have a lower (but for E<<1 still very large) critical Rayleigh number.

Symbols: V^T – toroidal velocity amplitude, H – helicity. .

Discussion of linear stability results IV

8) What changes when rotation and gravity vector or not parallel ? Assume $\mathbf{g} = -g\mathbf{e}_z$ and $\mathbf{\Omega} = \Omega\,(\sin\beta\;\mathbf{e}_z + \cos\beta\;\mathbf{e}_y)$.

In the case $\beta=0^o$ the preferred flow consists of 2D-rolls aligned with the y-direction. This flow is not in conflict with the Proudman-Taylor theorem. At onset of convection the flow is identical to that of the Rayleigh-Benard case and has the same critical Rayleigh number (657.5) and critical aspect ratio ($2^{1/2}$). The only differences to the case without rotation are that now the pattern (rolls) and their alignment are constrained and that a strong pressure gradient between the center of the roll and its margins balances the Coriolis force (purely geostrophic convection possible).

When $\beta > 0^o$ rolls aligned with the y-direction are still preferred. Assuming this particular flow pattern, the terms describing the effect of the Coriolis force on slide 6.5 now take the form $2E^{-1}\sin\beta\;\partial/\partial z$, which can be expressed by an effective Ekman number $E' = E/\sin\beta$ and the critical Rayleigh number and critical wavenumber depend on E' in the same way as derived before for $\beta=90^o$. Unless β is very small, there is again a strong inhibitation of convection by the effects of rotation.