Fundamentals of thermal convection I

Navier-Stokes-Equation (conservation of momentum) incompressible flow

\[ \rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) + \nabla p = \eta \nabla^2 \mathbf{u} + \mathbf{F}_{ext} \]

Inertial pressure viscous external force (conservation of mass) \( \nabla \cdot \mathbf{u} = 0 \)

Assumptions and approximations: incompressible flow (\( \rho = \text{const} \)), inertial (non-rotating) frame of reference, constant Newtonian viscosity. External forces considered are gravitational forces and electromagnetic forces.

Boundary conditions: (1) \( \mathbf{u} = 0 \) impenetrable no-slip boundary

(2) \( u_n = \frac{\partial \mathbf{u}}{\partial n} = 0 \) impenetrable free-slip boundary

Symbols (bold symbols denote vector): \( \rho \) – density, \( \mathbf{u} \) – velocity, \( p \) – pressure, \( \eta \) – dynamic viscosity, \( n \) – direction normal to boundary, \( \mathbf{n} \) – direction parallel to boundary
Boussinesq approximation

In thermal convection the flow is driven by differences in temperature that lead by thermal expansion to (usually small) differences in fluid density: \( \rho = \rho_o(1 - \alpha T) \). Conflict with assumption of incompressibility. Boussinesq approximation: assume \( \rho = \text{const} \) in all terms, except in that for the external gravity force: \( \mathbf{F}_{\text{ext}} = \rho \mathbf{g} = \rho_o \mathbf{g}(1 - \alpha T) \). \( \mathbf{g} \) given by gradient of potential: define hydrostatic pressure \( \nabla p_H = \rho_o \mathbf{g} \) and dynamic pressure \( P = p - p_H \).

\[
\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla P / \rho_o = \nu \nabla^2 \mathbf{u} - \mathbf{g} \alpha T
\]

Energy equation (heat transport equation). In the Boussinesq case, adiabatic heating and frictional heat are zero.

\[
\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + H'
\]

**Symbols** (index \( o \) denotes standard or reference value): \( \alpha \) – volumetric thermal expansion coefficient, \( T \) – temperature, \( g \) – gravity, \( P \) – dynamic pressure, \( \nu = \eta / \rho \) – kinematic viscosity, \( \kappa \) – thermal diffusivity, \( H' = H / (\rho c_p) \) – \( H \) is specific heat generation rate per unit volume and \( c_p \) is specific heat capacity.
Rayleigh – Bénard convection

Plane layer of height D and large (infinite) horizontal extent filled with a Newtonian fluid with constant material properties. Cartesian coordinates \( x,y,z, \ g = -g\ e_z \). Temperature fixed to \( T=T_o+\Delta T \) at \( z=0 \) and \( T=T_o \) at \( z=D, H'=0 \).

**Scaling of equations**

Non-dimensional variables: \((x',y',z') = (x,y,z)/D, t' = t \kappa/D^2, T'=(T-T_o)/\Delta T, u' = u \ D/\kappa, P' = P \ D^2/(\kappa \eta)\).

Non-dimensional equations (omitting primes):

\[
\frac{1}{Pr} \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla P = \nabla^2 u + Ra \ T \ e_z \\
\frac{\partial T}{\partial t} + u \cdot \nabla T = \nabla^2 T \quad \nabla \cdot u = 0
\]

Boundary conditions: \( w = \partial u/\partial z = \partial v/\partial z = 0 \) at \( z=0 \) and \( z=1 \); \( T(z=0) = 1; T(z=1) = 0 \).

By scaling, we replace seven physical parameters \((\alpha,\kappa,\nu,\Delta T,T_o,g,D)\) by two numbers.

**Symbols:** \( e_z \) – unit vector in z-direction (vertical), \( D \) – height of layer, \( Ra \) – Rayleigh number, \( Pr \) – Prandtl number, \( u = (u,v,w) \) – cartesian velocity components
Linear stability analysis (I)

- Trivial solution: \( T = 1 - z, \ u = 0, \ P = 0 \). Is this solution stable, i.e. will small perturbation decay?
- In Earth's mantle \( Pr >>1 \), assume \( Pr = \infty \).
- Assume 2-D solution (independence of \( y \), \( v=0 \)).
- Take curl of Navier-Stokes equation (eliminates pressure). Note that \( \nabla \times (Te_z) = -\partial T/\partial x \ e_y \).
- Represent 2D incompress. flow by stream function \( \psi \): \( u = (u,0,w) = \nabla \times (\psi e_y) = (\partial \psi/\partial z, 0, -\partial \psi/\partial x) \).
- Note that for any \( a \) with \( \nabla \cdot a = 0 \), \( \nabla^2 a = -\nabla \times (\nabla \times a) \). Operators \( \nabla^2 \) and \( \nabla \times \) commute. \( \omega = \nabla \times u \) is called vorticity.
- Perturbation \( T = 1-z+\theta, \ \theta << 1, \ u << 1 \). Ignore quadratic terms in small quantities.

\[
\nabla^4 \psi = Ra \frac{\partial \theta}{\partial x} \quad \quad \frac{\partial \theta}{\partial t} + \frac{\partial \psi}{\partial x} = \nabla^2 \theta
\]

- Boundary conditions for \( \psi \) and \( \theta \): \( \psi = \partial^2 \psi/\partial z^2 = \theta = 0 \) at \( z=0 \) and \( z=1 \).
- Expand into normal modes in \( x \)-direction: \( \theta = \theta_k(z,t) \exp(ikx); \ \psi = \psi_k(z,t) \exp(ikx) \).
- Expand in harmonic function in \( z \)-direction: \( \theta_k(z,t) = \theta_{kn}(t) \sin(n\pi z), \ \psi_k(z,t) = \psi_{kn}(t) \sin(n\pi z) \).

Note that sine-functions satisfy all the boundary conditions.

Symbols: \( \psi \) – stream function, \( \theta \) – temperature perturbation, \( k \) – horizontal wave number
Linear stability analysis (II)

\[(k^2 + n^2 \pi^2)^2 \psi_{kn} = Ra \, ik \, \theta_{kn}\]
\[d\theta_{kn}/dt = -ik \, \psi_{kn} - (k^2 + n^2 \pi^2) \theta_{kn}\]

Eliminate \(\psi_{kn}\):
\[\frac{d\theta_{kn}}{dt} = \left( k^2 + n^2 \pi^2 \right) \left( \frac{k^2 \, Ra}{(k^2 + n^2 \pi^2)^3} - 1 \right) \theta_{kn} = \sigma \, \theta_{kn}\]

The solution has the form \(\theta_{kn} \sim e^{\sigma t}\). When, at a given value of \(Ra\), \(\sigma < 0\) for all \(n\) and all \(k\), the trivial (conductive) solution is stable. When for some \(k\) and \(n\) \(\sigma > 0\), the conductive solution is unstable and convection will start. The critical Rayleigh number \(Ra_{crit}\) is found by seeking the lowest \(Ra\) for which \(\sigma = 0\) is reached at any \(k,n\). Obviously, the minimum is obtained for \(n=1\).

\[Ra_c(k) = \frac{(k^2 + \pi^2)^3}{k^2}\]

Minimum at \(k_{crit} = \pi/\sqrt{2}\). Wavelength is \(\lambda = 2 \sqrt{2}\).
Width of one convection cell (aspect ratio) is \(\sqrt{2}\).

\[Ra_{crit} = Ra_c(k_{crit}) = 27\pi^4/4 \approx 657.5\]

Symbols: \(\sigma\) - growth rate, \(Ra_{crit}\) - critical Rayleigh number, \(k_{crit}\) - critical wave number
Linear stability analysis (III)

• Result unchanged when 3-D convection pattern is allowed. The critical wave number is then $|k| = (k_x^2 + k_y^2)^{1/2} = 2\sqrt{2}$. Linear stability cannot discriminate between different planforms of convection, which is controlled by the non-linear terms. At small super-critical Rayleigh number, the preferred pattern is two-dimensional (convection rolls).

• Result is unchanged if a finite value of Pr is retained.

• Similar analysis (although more complicated) for other boundary conditions. For example with no slip boundaries $R_{\text{crit}} = 1708$.

• In the case of internal heating, $H > 0$, $\partial T/\partial z = 0$ at $z=0$, $T=0$ at $z=1$, the Rayleigh number must be re-defined, replacing $\Delta T$ by the characteristic temperature contrast in the conductive state, $HD^2/k$

$$R_{a_H} = \frac{\alpha g HD^5}{k \kappa \nu}$$

The critical Rayleigh number in this case is, with free-slip boundaries, $R_{\text{crit}} = 868$.

• In a broad range of other cases (various combinations of mechanical and thermal boundary conditions, spherical geometry) the critical Rayleigh number is typically of the order $10^3$.

Symbols: $k_x, k_y$ – components of wavenumber vector, $k$ – thermal conductivity,
Pattern of convection at $Ra > Ra_{\text{crit}}$

Visualization of pattern in convection experiments by shadowgraph technique. Regular geometrical pattern are observed at moderate values of the Rayleigh number, up to $\approx 10\ Ra_{\text{crit}}$.

At larger Rayleigh number the flow becomes irregular and time-dependent.

Symbols: $k_x, k_y$ – components of wavenumber vector, $k$ – thermal conductivity,
Application to Earth and Planets

For Earth’s mantle select characteristic values:

\[
\begin{align*}
\alpha &= 2 \times 10^{-5} \text{ K}^{-1} \\
\Delta T &= 2000 \text{ K} \\
g &= 10 \text{ m/s}^2 \\
D &= 2,900,000 \text{ m} \\
\kappa &= 10^{-6} \text{ m}^2/\text{s} \\
\rho &= 4000 \text{ kg/m}^3 \\
\eta &= 10^{21} – 10^{22} \text{ Pa s} \quad \text{(from postglacial rebound)}
\end{align*}
\]

\[\implies Ra = 4 \times 10^6 \ldots 4 \times 10^7 \gg Ra_{\text{crit}}\]

For other planets, assume similar values for \(\alpha, \rho, \kappa, \eta\). Use \(Ra_H\) with the „chondritic“ value of radiogenic heating \(H = 1.6 \times 10^{-8} \text{ Wm}^{-3}\) (\(H' = 4 \times 10^{-15} \text{ K/s}\)). Without plate tectonics convection takes place below a rigid outer shell whose bottom at radius \(r_*\) is given by a temperature of \(T_* \approx 1300 \text{ K}\) (heat is conducted in the shell). In a conducting, internally heated sphere

\[T(r) = H'/(6\kappa) [r_o^2-r^2] + T_o \implies r_*^2 = r_o^2 - 6\kappa(T_*-T_o)/H'. \quad \text{Set } D = r_* - r_c.\]

<table>
<thead>
<tr>
<th></th>
<th>(r_o) [km]</th>
<th>(r_c) [km]</th>
<th>(T_o) [K]</th>
<th>(r_*) [km]</th>
<th>(g_o) [m/s²]</th>
<th>(Ra_H)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Venus</td>
<td>6050</td>
<td>3200</td>
<td>720</td>
<td>5950</td>
<td>8.9</td>
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<tr>
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<td>3400</td>
<td>1500</td>
<td>220</td>
<td>3060</td>
<td>3.7</td>
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<tr>
<td>Moon</td>
<td>1740</td>
<td>400</td>
<td>250</td>
<td>960</td>
<td>1.6</td>
<td>3 \times 10^4</td>
</tr>
</tbody>
</table>

Symbols: \(r_o\) – outer radius of planet, \(T_o\) – temperature at \(r_o\), \(T_*\) - transitional temperature, \(r_*\) - radius of lithosphere-asthenosphere boundary, \(r_c\) – core radius, \(g_o\) – gravity at surface