2. STABILITY THEORY

1. Equilibrium states

When analyzing physical systems, we often start by seeking steady (time-independent) states.

**Examples**

![Diagram of equilibrium states](image)

Steady (equilibrium) states are possible solutions of the full time-dependent equations governing the evolution of the system.

But steady states may be **unstable**. That is, any small perturbation (deviation) from the equilibrium state will grow in amplitude.

Such a state is therefore a highly improbable configuration of the physical system.

2. Stability of a Simple Pendulum

(System with one degree of freedom)

![Diagram of a pendulum](image)

\[
ml^2 \frac{d^2 \theta}{dt^2} = -mgl \sin \theta - kl \frac{d\theta}{dt}
\]

acceleration    restoring moment    viscous damping

2.1

2.2
Dimensional analysis

Aim to determine the dimensionless groups of parameters upon which the behaviour of the system depends.

\[ [m] = M, \quad [l] = L, \quad [g] = \frac{L}{T^2}, \quad [k] = \frac{ML}{T} \]

There are two independent time scales

\[ t_1 = \sqrt{\frac{T}{g}}, \quad t_2 = \frac{ml}{k} \]

Choose to write \( t = \sqrt{\frac{T}{g}} \tau \) so that \( \tau \) is a dimensionless variable.

Then

\[ \frac{d^2 \theta}{d \tau^2} + \frac{\kappa}{m l g} \frac{d \theta}{d \tau} + \sin \theta = 0 \]

where \( \kappa = \frac{k}{m l g} = \frac{t_1}{t_2} \), the ratio of time scales, is the only parameter governing the evolution of the system.

Equilibrium states

\[ \frac{d \theta}{d \tau} = 0 \Rightarrow \sin \theta = 0 \Rightarrow \theta = \theta_0 = 0, \pi \]

Equilibrium is independent of \( \kappa \).

Perturbation

\[ \theta = \theta_0 + \varepsilon(\tau) \Rightarrow \ddot{\varepsilon} + \kappa \varepsilon + [\cos \theta_0] \sin \varepsilon = 0 \]

Linearization

\[ \varepsilon << 1 \Rightarrow \sin \varepsilon = \varepsilon \Rightarrow \ddot{\varepsilon} + \kappa \varepsilon + [\cos \theta_0] \varepsilon = 0 \]

This is a linear equation with constant coefficients

\[ \Rightarrow \varepsilon \approx e^{\alpha \tau} \]

\[ \Rightarrow \sigma^2 + \kappa \sigma + \cos \theta_0 = 0 \]

\[ \Rightarrow 2\sigma = -\kappa \pm \sqrt{\kappa^2 - 4 \cos \theta_0} \]

For each value of \( \theta_0 \), there are two values of \( \sigma \) (\( \sigma_1 \) and \( \sigma_2 \) say).
General solution is

\[ \varepsilon = Ae^{\sigma_1 t} + Be^{\sigma_2 t} \]

1. \( \theta_0 = 0 \Rightarrow 2\sigma = -\kappa \pm \sqrt{\kappa^2 - 4} \)
   \( \sigma_1, \sigma_2 \) both negative \( \Rightarrow \varepsilon \to 0 \text{ as } t \to \infty \)
   System is **STABLE**

2. \( \theta_0 = \pi \Rightarrow 2\sigma = -\kappa \pm \sqrt{\kappa^2 + 4} \)
   One of \( \sigma_1, \sigma_2 \) is positive \( \Rightarrow \varepsilon \to \infty \text{ as } t \to \infty \)
   System is **UNSTABLE**

3. Stability of a Double Pendulum
   (System with two degrees of freedom)

   ![Diagram of a double pendulum]

   \[ \ddot{\theta}_1 + \frac{1}{2} \cos(\theta_2 - \theta_1) \dot{\theta}_2^2 - \frac{1}{2} \sin(\theta_2 - \theta_1) \dot{\theta}_1 \dot{\theta}_2 + \sin \theta_1 = 0 \]
   \[ \ddot{\theta}_2 + \cos(\theta_2 - \theta_1) \dot{\theta}_1^2 - \sin(\theta_2 - \theta_1) \dot{\theta}_1 \dot{\theta}_2 + \sin \theta_2 = 0 \]

   Consider equilibrium state \( \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} = \begin{pmatrix} \pi \\ 0 \end{pmatrix} \)

   Perturb steady state \( \theta_1 = \pi + \varepsilon_1(t) \quad \theta_2 = \varepsilon_2(t) \)

   Linearized perturbation equations \( \ddot{\varepsilon}_1 - \frac{1}{2} \dot{\varepsilon}_2 - \varepsilon_1 = 0 \quad \ddot{\varepsilon}_2 - \dot{\varepsilon}_1 + \varepsilon_2 = 0 \)
In general, $\epsilon_1$ and $\epsilon_2$ are different functions of time. We can find special solutions, called normal modes, in which $\epsilon_1$ and $\epsilon_2$ have the same time dependence.

$$\epsilon_1 = af(t) \quad \text{where} \quad a, b \quad \text{are constants.}$$

$$\epsilon_2 = bf(t)$$

Normal modes have the property that the shape or configuration of the system doesn't change; only the amplitude changes with time.

**Example**

The ratio $\frac{\epsilon_1}{\epsilon_2}$ remains constant.

Since perturbation equations are linear with constant coefficients

$$f(t) = e^{\sigma t} \quad \text{where} \quad \sigma \quad \text{is the growth rate of the mode.}$$

Substitute normal-mode solutions into perturbation equations to obtain

$$\sigma^2 a - \frac{1}{2} \sigma^2 b - a = 0$$

$$\sigma^2 b - \frac{1}{2} \sigma^2 a + b = 0$$

which can be written in matrix form as

$$\begin{pmatrix} \sigma^2 - \frac{1}{2} \sigma^2 & \frac{1}{2} \\ -\sigma^2 & \sigma^2 + 1 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = 0 \quad (\star)$$

This has non-zero solutions only if the determinant of the matrix is zero

$$\Rightarrow \sigma^4 - \frac{1}{2} \sigma^4 = 0$$

$$\Rightarrow \sigma^4 = \pm \sqrt{2}$$
1. \( \sigma^2 = +\sqrt{2} \Rightarrow f(t) = Ce^{\sqrt{2}t} + De^{-\sqrt{2}t} \)

\[ f(t) \to \infty \text{ as } t \to \infty \text{ so mode is UNSTABLE} \]

From (*) \[ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2-\sqrt{2} \end{pmatrix} \]

2. \( \sigma^2 = -\sqrt{2} \Rightarrow f(t) = E \sin \frac{\sqrt{2}}{2} t + F \cos \frac{\sqrt{2}}{2} t \)

\[ f(t) \text{ bounded as } t \to \infty. \text{ Mode is STABLE} \]

From (*) \[ \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 1 \\ 2+\sqrt{2} \end{pmatrix} \]

5. Superposition of Normal Modes

A general solution \( \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix} \) can be written as a sum of normal modes.

For example

\[ \begin{pmatrix} \varepsilon_1(t) \\ \varepsilon_2(t) \end{pmatrix} = \begin{pmatrix} \varepsilon_G(t) \\ \varepsilon_B(t) \end{pmatrix} - \begin{pmatrix} \varepsilon_A(t) \\ \varepsilon_A(t) \end{pmatrix} \]

Since the amplitude of mode \( \mathbf{A} \) grows in time while the amplitude of mode \( \mathbf{B} \) remains constant, every configuration of the system evolves to look more and more like mode \( \mathbf{A} \).

When a system is perturbed from an unstable equilibrium, one tends to see the mode with the largest growth rate.
(System with infinite degrees of freedom.)

Notice from demonstration:
1. A specific spatial structure (wavelength) evolves from random disturbances.
2. Instability of one steady state may lead to another (stable) steady state.

Mode (wavelength) selection

Random disturbances can be expressed as a superposition of pure sinusoidal disturbances (Fourier modes), which are the normal modes of this system.

There are an infinite number of normal modes - one for each value of the wavenumber

$$k = \frac{2\pi}{\text{wavelength}}$$

Analysis reveals the growth rate $\sigma(k)$ for each normal mode.

All modes with $k < \frac{1}{a}$ (wavelength $> 2\pi a$) are unstable, where $a$ is the radius of the undisturbed cylinder.

Will tend to see disturbances with wavenumber $= k_{\text{max}}$

N.B. Dimensional analysis gives

$$\sigma \approx \frac{\gamma}{\mu a}$$

$\mu$: viscosity

$\gamma$: surface tension
7. Rayleigh-Taylor Instability of Superposed Fluids

Dense fluid ($\rho_2$) lies above lighter fluid ($\rho_1$) in a square cylinder of side length $a$. Steady state has interface flat at $z=0$ and no flow.

Small disturbances described by interface at position $z = \eta(x, y, t)$ and fluid velocity $u = \nabla \eta$ (inviscid, irrotational)

Then $v^2\phi = 0 \quad \phi \to 0$ as $z \to \pm\infty$

\[
\frac{\partial \phi}{\partial x} = 0 \quad (x = 0, a) \quad \text{kinematic boundary conditions}
\]

\[
\frac{\partial \phi}{\partial y} = 0 \quad (y = 0, a)
\]

\[
\frac{\partial \eta}{\partial z} = 0 \quad (z = \eta)
\]

\[
\rho_1 - \rho_2 = -\gamma \mathbf{v} \cdot \mathbf{n} \quad (z = \eta) \quad \text{dynamic b.c.}
\]

where $\mathbf{n}$ is normal to interface, from fluid 1 to fluid 2, and the pressure is found from Bernoulli. $p + \rho \frac{\partial \phi}{\partial t} + \rho g \eta + \frac{1}{2} \rho (\nabla \eta)^2 = 0$

8. Scaling and Linearization

\[
(x, y, z) = a(x^*, y^*, z^*) \quad , \quad t = \sqrt{\frac{\rho_2 a^3}{\gamma}} t^* \quad , \quad \mathbf{q} = \sqrt{\frac{\rho_2}{\gamma}} \mathbf{q}^*
\]

The starred variables are dimensionless. Substitute into equations, linearize with $\eta \ll 1$, $\phi \ll 1$ and drop stars.

N.B. In linear problem, interfacial conditions are applied at $z = 0$.

\[
\nabla^2 \phi = 0 \quad \phi \to 0 \quad (z \to \pm\infty)
\]

\[
\frac{\partial \phi}{\partial x} = 0 \quad (x = 0, a)
\]

\[
\frac{\partial \phi}{\partial y} = 0 \quad (y = 0, a)
\]

\[
\frac{\partial \phi_1}{\partial z} = \frac{\partial \phi_2}{\partial z} \quad (z = 0)
\]

\[
\beta \phi_{2t} - \phi_{2tt} = \left(R + \nabla^2 \eta \right) \phi_2 \quad (z = 0)
\]

where $\beta = \frac{\rho_1}{\rho_2}$, $\gamma = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ and $R = \frac{(\rho_2 - \rho_1)g a^2}{\gamma}$, the Bond number, is a ratio of the time scales $t_1 = \sqrt{\frac{\rho a^3}{\gamma}}$ and $t_2 = \sqrt{\frac{a}{g'}} \left[ g' = \frac{\rho_2 - \rho_1}{\rho_2} g \right]$. 


Normal modes

\[ \varphi_{1,2} = \pm e^{i \xi} e^{\pm i \xi} \cos n \pi x \cos m \pi y \]

\[ \Rightarrow (1 + \beta) \sigma^2 = (R - k^2)k \]

The system is unstable if

\[ R > k^2 \Rightarrow a^2 > \left( n^2 + m^2 \right) \pi^2 \gamma \frac{1}{(\rho_2 - \rho_1)g} \]

For water over air:

unstable if \( a > 8.6 \) mm

9. Shear-Flow Instability

**Film Demonstration - Mollo Christiansen**

**THE NEUTRAL CURVE**

For \( U > U_{\text{min}} \), there is a band of unstable modes (wavelengths). Linear theory only tells us that disturbances grow. What they grow into is the subject of nonlinear analyses.
Lecture 2. Stability Theory