ロスビー波

石渡正樹

2020 年 06 月 15 日

第1章　ロスビー波とは

1.1　ロスビー波とは

ロスビー波とはポテンシャル渦度保存則に根をもつ波動擾乱の総称名称である．ポテンシャル渦度保存則が形式的に

$$\frac{\partial q}{\partial t} + \boldsymbol{v} \cdot \nabla q = 0 \tag{1.1}$$

と書き下されているものとしよう．ここで, q はポテンシャル渦度, $\boldsymbol{v}$ は流速場, (t, $\boldsymbol{x}$) は時間空間座標である．ポテンシャル渦度と流速場との間に時間微分を含まない適当な作用素 $\mathcal{L}$ が (近似的に) 存在して

$$\boldsymbol{v} = \mathcal{L}(q) \tag{1.2}$$

と書き表されるとき, (1.1) は q の t に関する 1 階の偏微分方程式となる．得られた方程式はその波動解を議論する際にはロスビー波の式と呼ばれる．

地球流体で議論される線型波動擾乱としてのロスビー波はポテンシャル渦度が $Q(y)$, 流れが $U(y)$ である基本場に対する微小擾乱

$$\left\{ \begin{array}{lll} q = & Q+ & q' \\ u = & U+ & u' \\ v = & & v' \end{array} \right.$$

として現れる．このような状況下でのポテンシャル渦度保存則は

$$\frac{\partial q'}{\partial t} + U\frac{\partial q'}{\partial x} + \frac{dQ}{dy}v = 0.$$

である (鉛直移流項がなくなっていることに注意).

以下では, 成層・粘性のない β 面 2 次元シアー流中の微小擾乱を考える．このような状況はロスビー波のもっとも簡単な描像を与える．

第2章　2 次元非発散系におけるロスビー波

2.1　基礎方程式, 線型化

系として β 面上の 2 次元非発散系を考察する．支配方程式は次のように書きくださせる．

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.1}$$

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{du}{dy}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.2}$$

$$\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{dv}{dy}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{2.3}$$

ただし $f(y)=f_0+\beta y$, $\rho=\mathrm{const.}$ である．

基本場が $u=U(y)$ である流れに対する線型化された擾乱の方程式は,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \tag{2.4}$$

$$\frac{\partial u}{\partial t}+U\frac{\partial u}{\partial x}+v\frac{dU}{dy}-fv=-\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.5}$$

$$\frac{\partial v}{\partial t}+U\frac{\partial v}{\partial x}+fu=-\frac{1}{\rho}\frac{\partial p}{\partial y}. \tag{2.6}$$

(2.1) より流線関数 $u=-\dfrac{\partial\psi}{\partial y}$, $v=\dfrac{\partial\psi}{\partial x}$ を導入することができる．(2.4)∼ (2.6) より渦度方程式を作ると

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi-\frac{d^2U}{dy^2}\frac{\partial\psi}{\partial x}+\beta\frac{\partial\psi}{\partial x}=0. \tag{2.7}$$

この方程式は (1.1) において基本場のポテンシャル渦度は $Q(y)=f(y)-\dfrac{dU}{dy}$, 擾乱のポテンシャル渦度は $q'=\nabla^2\psi$, 速度 $v=\dfrac{\partial\psi}{\partial x}$ とした場合に相当する．

特に $U(y) \equiv 0$ のとき, 線型化された方程式は

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \tag{2.8}$$

$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2.9}$$

$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} \tag{2.10}$$

また, (2.7) は次のようになる.

$$\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi + \beta\frac{\partial \psi}{\partial x} = 0. \tag{2.11}$$

2.2　分散関係, 群速度

以下, ロスビー波をつかさどるもっとも単純な方程式 (2.11) を取り上げ, ロスビー波のイメージをつかむことにする.

2.2.1　分散関係

分散関係式を求めるために解として平面波の形 $\psi = \psi_0 e^{i(kx+ly-\omega t)}$ を (2.11) に代入し整理すると

$$\omega = -\frac{\beta k}{k^2 + l^2}. \tag{2.12}$$

これが 2 次元非発散ロスビー波の分散関係である.

2.2.2　位相速度

$$c_{px} \equiv \frac{\omega}{k} = -\frac{\beta}{k^2 + l^2}, \tag{2.13}$$

$$c_{py} \equiv \frac{\omega}{l} = -\frac{\beta k}{l(k^2 + l^2)}. \tag{2.14}$$

$c_{px} < 0$ より, 位相は常に x 軸負方向 (西向き) に進む.

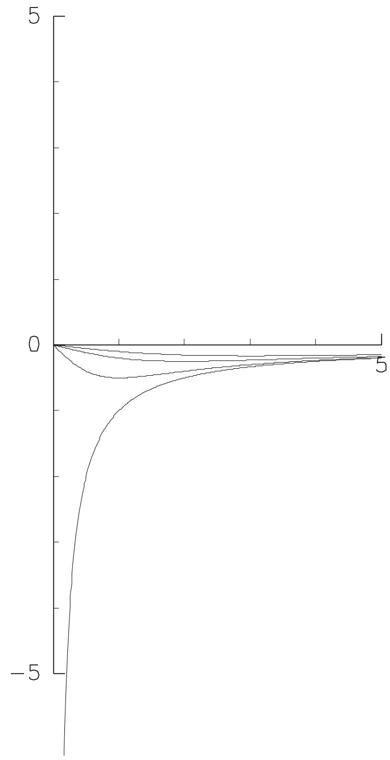

図 2.1: 2 次元非発散ロスビー波の分散関係 ($k-\omega$ 面). $l=0,1,2,3$ の場合を示す. 横軸は k, 縦軸は σ.

2.2.3 群速度

$$c_{gx} \equiv \frac{\partial \omega}{\partial k} = \frac{\beta(k^2-l^2)}{(k^2+l^2)^2}, \tag{2.15}$$

$$c_{gy} \equiv \frac{\partial \omega}{\partial l} = \frac{2\beta kl}{(k^2+l^2)^2}. \tag{2.16}$$

$k<l$ の波束は x 軸負方向 (西向き), $k>l$ の波束は x 軸正方向 (東向き) にエネルギーを伝播する.

2.2.4 $k-l$ 面での分散関係の表現

(2.12) を変型することにより

$$\left(k+\frac{\beta}{2\omega}\right)^2 + l^2 = \frac{\beta}{4\omega^2}. \tag{2.17}$$

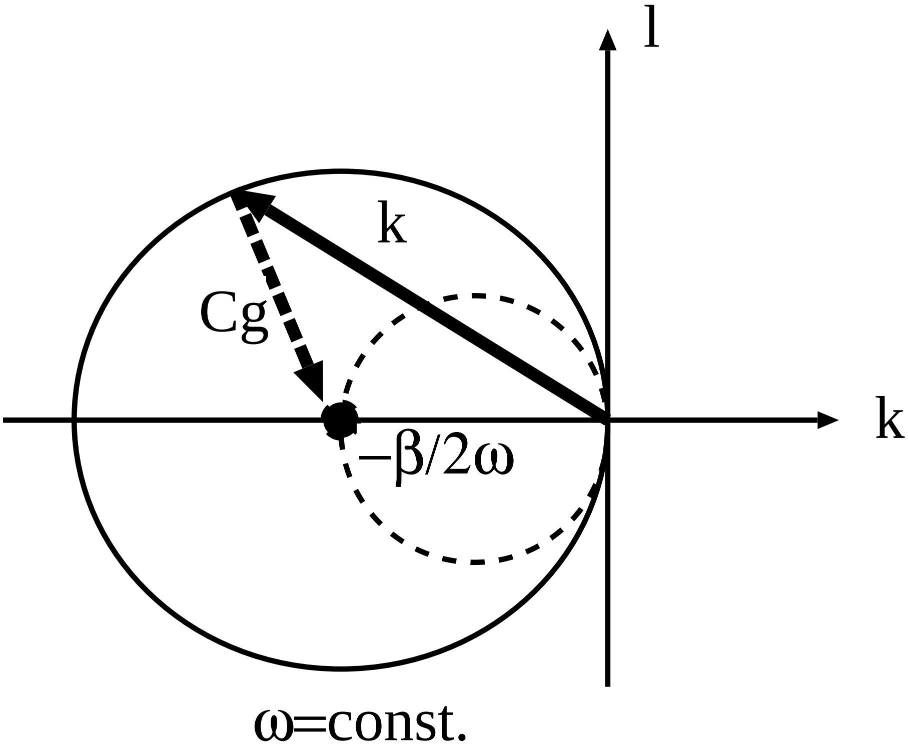

図 2.2: ロスビー波の分散関係 ($k-l$ 面)

ω =const. である k,l を $k-l$ 面で表すと円になる (図 2). また群速度は $k-l$ 面での ω の勾配 ($\boldsymbol{c}_g = \nabla_k\omega$, $\nabla_k = \boldsymbol{i}\frac{\partial}{\partial k} + \boldsymbol{j}\frac{\partial}{\partial l}$) であるから, その向きは (2.17) の円周上の点から円の中心に向かう向きとなる.

2.3　初期値問題 ∼ ロスビー波の分散

2次元 β 面内に初期擾乱を与えたときの時間変化の計算例を図 2.3 に示す. 与えられた擾乱がロスビー波として伝播し, 分散していく.

図 2.3 (c),(d) において図 2.4 のような位相と群速度の関係が見られる.

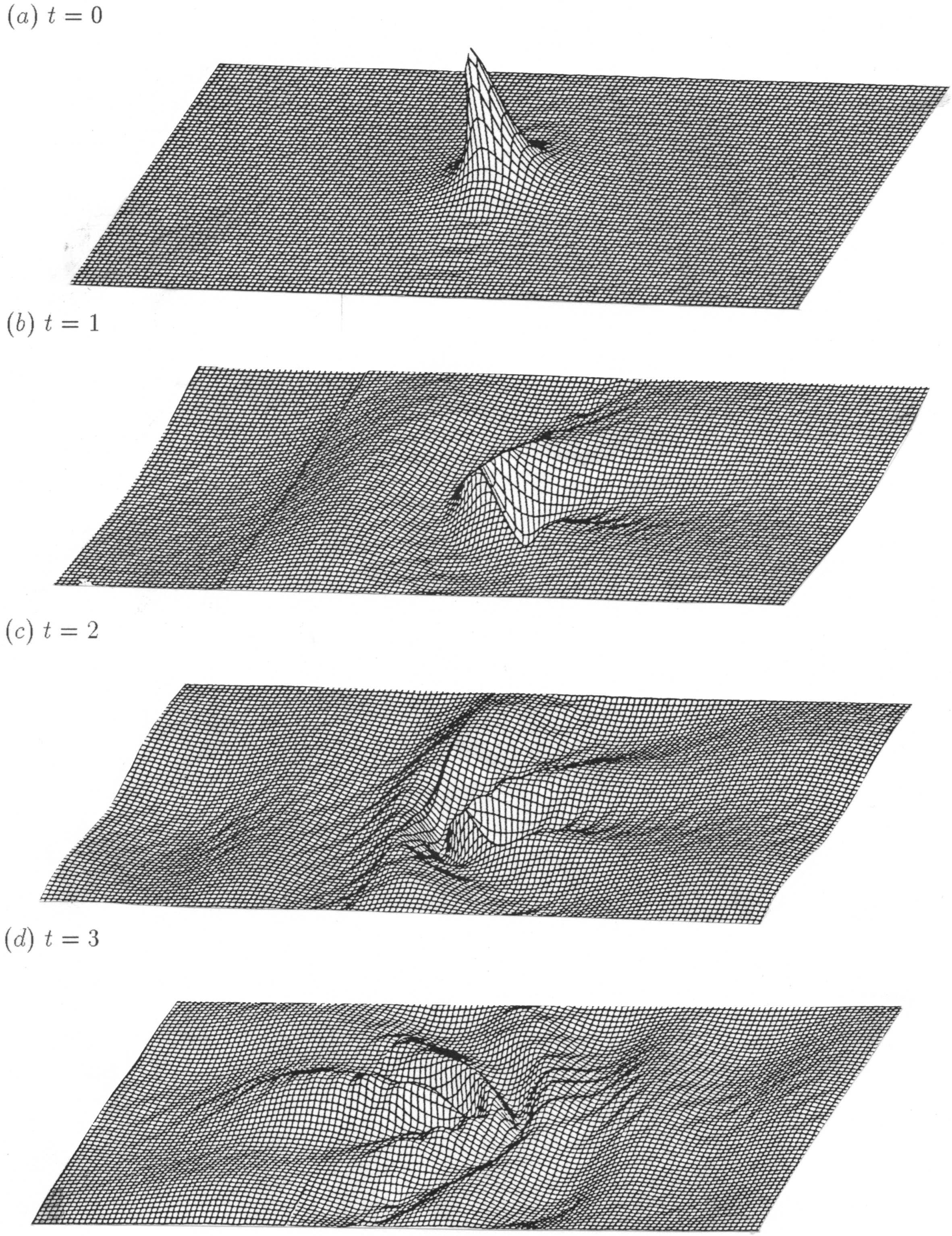

図 2.3: 初期擾乱を与えたときの位相と群速度の関係.

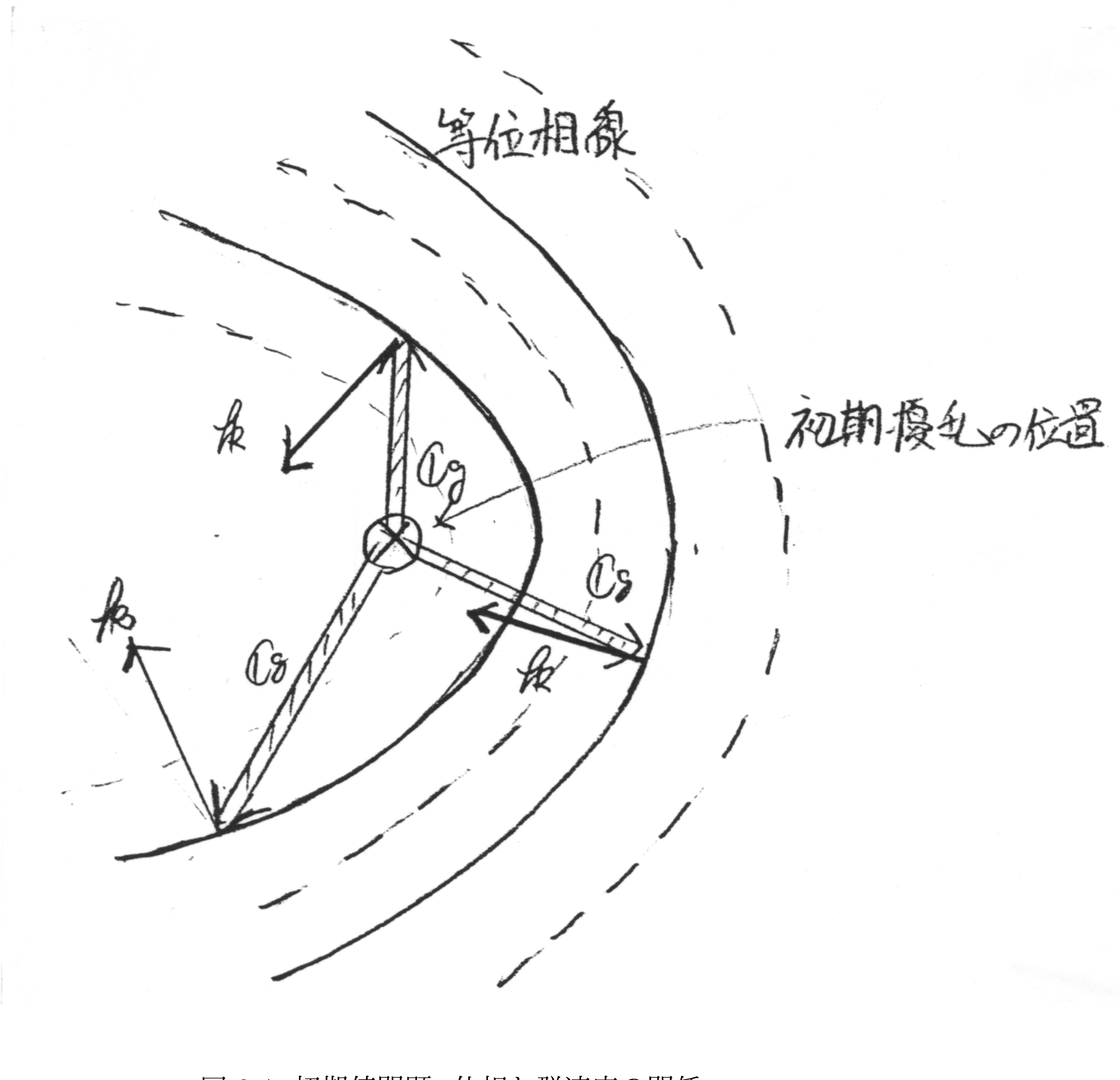

図 2.4: 初期値問題. 位相と群速度の関係.

2.4 ロスビー波の構造

相対渦度 ζ, 速度 u,v と流線関数の関係は

$$\begin{cases} u = -\dfrac{\partial \psi}{\partial y} = -ik\psi, \\ \\ v = \dfrac{\partial \psi}{\partial x} = il\psi, \\ \\ \zeta = \nabla^2 \psi = -(k^2 + l^2)\psi, \end{cases}$$

である. これより u,v,ζ の位相関係は, 図 2.5 のようになる.

渦度に対し, 波により作られる流れは 1/4 波長ずれている.

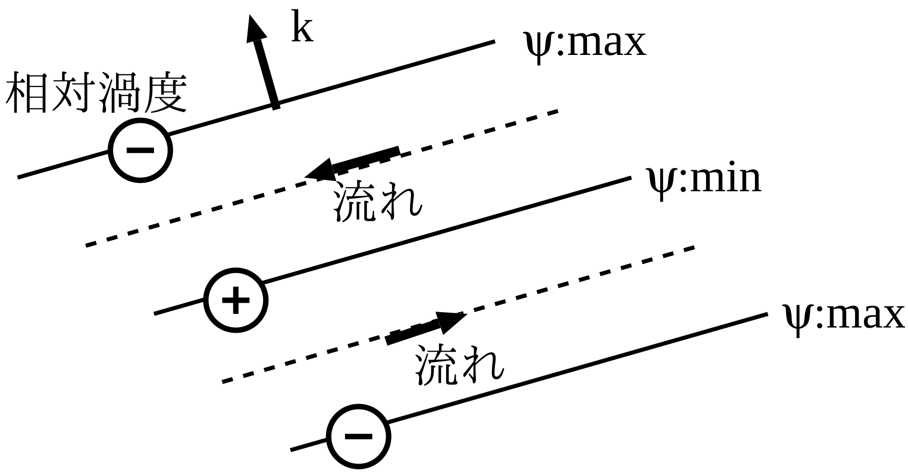

図 2.5: ロスビー波の構造.

2.5 ロスビー波の伝幡

ロスビー波の伝幡を, ポテンシャル渦度の保存

$$\frac{d}{dt}(f + \zeta) = 0, \quad f = f_0 + \beta y$$

により説明する.

2.5.1 Parcel的考え方

1. 初期状態で $y=$ 一定 の線上にある流体コラム A,B,C を考える (図3.2a). 今, B が y 軸正方向に微小変位したとする.
2. y 軸正方向に変位すると, f は増加する. ポテンシャル渦度が保存する には ζ は減少しなければならない. したがって y 軸正方向に変位した B は負の相対渦度を持つことになる. (図3.2b)
3. B の持つ渦度に伴って A では y 軸正方向, C では負方向の流れが引き起こされ, 変位する (図3.2c).
4. 2. と同様, 変位した A,C はそれぞれ負, 正の相対渦度を持つ (図3.2d).
5. A,C の持つ渦度に伴って, B では y 軸負方向の流れが引き起こされ, B は元の位置に向かって変位する (図3.2e).

この議論より, 場の (ポテンシャル) 渦度に勾配がなければ ($\beta=0$ or $\dfrac{d^2U}{dy^2}=0$ etc.) 擾乱の渦は伝幡しないことがわかる. この時の渦度方程式は

$$\frac{\partial}{\partial t}\nabla^2\psi=0,$$

すなわち分散関係は $\omega=0$ となり, 渦は伝幡せず動かない.

2.6 エネルギーの伝播

2.6.1 波線理論, WKB 近似を用いた記述

微小パラメータ ε を導入して (2.11) とその解 ψ を漸近的に展開して考える. ψ が次のように展開されるものとしよう:

$$\psi=\sum_{n=0}\varepsilon^n A_n(\boldsymbol{X},T)e^{i\Theta(\boldsymbol{x},t)/\varepsilon}. \tag{2.18}$$

ただし,

$$\boldsymbol{X}=\varepsilon\boldsymbol{x}, \tag{2.19}$$

$$T=\varepsilon t \tag{2.20}$$

はゆっくり変化する座標である.

(2.18) を (2.11) に代入して, $O(\varepsilon^0)$, $O(\varepsilon^1)$ の項を整理すると次のようになる.

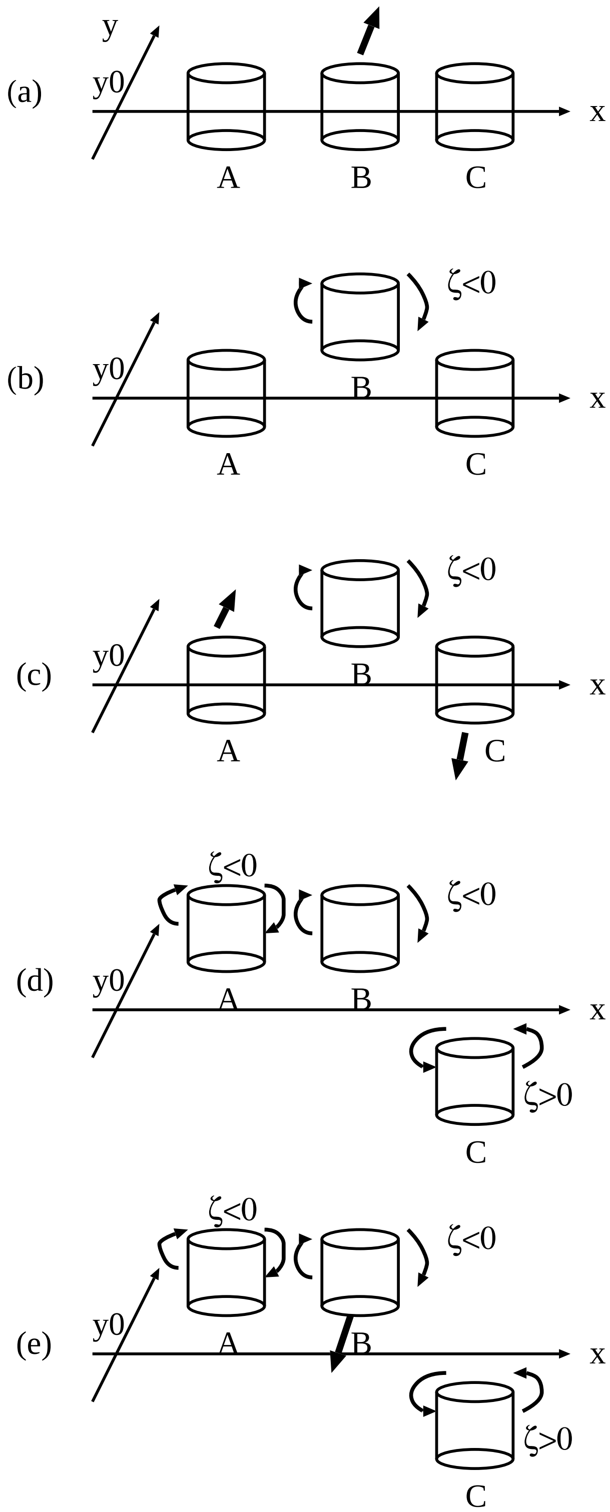

図 2.6: ロスビー波の伝幡

- $O(\varepsilon^0)$ の式

$$\left[-i\frac{\partial\Theta}{\partial T}\left(\frac{\partial\Theta}{\partial X}\right)^2 - i\frac{\partial\Theta}{\partial T}\left(\frac{\partial\Theta}{\partial Y}\right)^2 + i\beta\frac{\partial\Theta}{\partial X}\right]A_0 = 0. \tag{2.21}$$

- $O(\varepsilon^1)$ の式

$$\begin{aligned}&\left[-i\frac{\partial\Theta}{\partial T}\left(\frac{\partial\Theta}{\partial X}\right)^2 - i\frac{\partial\Theta}{\partial T}\left(\frac{\partial\Theta}{\partial Y}\right)^2 + i\beta\frac{\partial\Theta}{\partial X}\right]A_1 \\ &+\left[-2\frac{\partial^2\Theta}{\partial T\partial X}\frac{\partial\Theta}{\partial X} - \frac{\partial^2\Theta}{\partial X^2}\frac{\partial\Theta}{\partial T} - 2\frac{\partial^2\Theta}{\partial T\partial Y}\frac{\partial\Theta}{\partial Y} - \frac{\partial^2\Theta}{\partial Y^2}\frac{\partial\Theta}{\partial T}\right]A_0 \\ &-\left[\left(\frac{\partial\Theta}{\partial X}\right)^2 + \left(\frac{\partial\Theta}{\partial Y}\right)^2\right]\frac{\partial A_0}{\partial T} - 2\frac{\partial\Theta}{\partial X}\frac{\partial\Theta}{\partial T}\frac{\partial A_0}{\partial X} - 2\frac{\partial\Theta}{\partial Y}\frac{\partial\Theta}{\partial T}\frac{\partial A_0}{\partial Y} + \beta\frac{\partial A_0}{\partial X} = 0.\end{aligned} \tag{2.22}$$

局所振動数, 波数

$$(k, l) \equiv \left(\frac{\partial\Theta}{\partial X}, \frac{\partial\Theta}{\partial Y}\right), \tag{2.23}$$

$$\omega \equiv -\frac{\partial\Theta}{\partial T} \tag{2.24}$$

を導入して整理すると

- $O(\varepsilon^0)$ の式

$$\omega(k^2 + l^2) + \beta k = 0. \tag{2.25}$$

- $O(\varepsilon^1)$ の式

$$\begin{aligned}&\left[-2\frac{\partial k}{\partial T}k + \frac{\partial k}{\partial X}\omega - 2\frac{\partial l}{\partial T}l + \frac{\partial l}{\partial Y}\omega\right]A_0 \\ &\quad -(k^2 + l^2)\frac{\partial A_0}{\partial T} + 2k\omega\frac{\partial A_0}{\partial X} + 2l\omega\frac{\partial A_0}{\partial Y} + \beta\frac{\partial A_0}{\partial X} = 0.\end{aligned} \tag{2.26}$$

  ただし (2.6.1) を使って簡略化してある.

$O(\varepsilon^0)$ の式 (2.6.1) はロスビー波の分散関係式にほかならない. $O(\varepsilon^1)$ の式 (2.26) はロスビー波の振幅の変化を記述する式である.

2.6.2 付録: 漸近展開の式の導出

ここでは, (2.6.1), (2.26) の導出過程を記す.

まず準備としていくつかの公式を用意しておく.

$$\frac{\partial A_n}{\partial x} = \frac{\partial A_n}{\partial X}\frac{\partial X}{\partial x} = \epsilon\frac{\partial A_n}{\partial X}, \tag{2.27}$$

$$\frac{\partial \Theta}{\partial x} = \frac{\partial \Theta}{\partial X}\frac{\partial X}{\partial x} = \epsilon\frac{\partial \Theta}{\partial X}, \tag{2.28}$$

$$\frac{\partial}{\partial x}e^{\frac{i\Theta}{\epsilon}} = e^{\frac{i\Theta}{\epsilon}}\frac{1}{\epsilon}\frac{\partial \Theta}{\partial x} = ie^{\frac{i\Theta}{\epsilon}}\frac{\partial \Theta}{\partial X}, \tag{2.29}$$

$$\begin{aligned}\frac{\partial^2}{\partial x^2}e^{\frac{i\Theta}{\epsilon}} &= \frac{\partial}{\partial x}\left(ie^{\frac{i\Theta}{\epsilon}}\frac{\partial \Theta}{\partial X}\right) = i\frac{\partial}{\partial x}e^{\frac{i\Theta}{\epsilon}}\frac{\partial \Theta}{\partial X} + ie^{\frac{i\Theta}{\epsilon}}\frac{\partial}{\partial x}\frac{\partial \Theta}{\partial X} \\ &= -e^{\frac{i\Theta}{\epsilon}}\left(\frac{\partial \Theta}{\partial X}\right)^2 + ie^{\frac{i\Theta}{\epsilon}}\epsilon\frac{\partial^2 \Theta}{\partial X^2},\end{aligned} \tag{2.30}$$

$$\frac{\partial A_n}{\partial t} = \frac{\partial A_n}{\partial T}\frac{\partial T}{\partial t} = \epsilon\frac{\partial A_n}{\partial T}, \tag{2.31}$$

$$\frac{\partial \Theta}{\partial t} = \frac{\partial \Theta}{\partial T}\frac{\partial T}{\partial t} = \epsilon\frac{\partial \Theta}{\partial T}, \tag{2.32}$$

$$\frac{\partial}{\partial t}e^{\frac{i\Theta}{\epsilon}} = e^{\frac{i\Theta}{\epsilon}}\frac{1}{\epsilon}\frac{\partial \Theta}{\partial t} = ie^{\frac{i\Theta}{\epsilon}}\frac{\partial \Theta}{\partial T} \tag{2.33}$$

これらを使って, 渦度方程式の各項の展開式を計算する. $O(\epsilon^0)$ の次数に応じて項を取りだすように計算する. 各項について計算すると以下のようになる.

$$\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\sum \varepsilon^n A_n e^{\frac{i\Theta}{\varepsilon}} = \sum \varepsilon^n \left[\epsilon\frac{\partial A_n}{\partial X}e^{\frac{i\Theta}{\varepsilon}} + A_n ie^{\frac{i\Theta}{\epsilon}}\frac{\partial \Theta}{\partial X}\right],$$

$$\begin{aligned}\frac{\partial^2 \psi}{\partial x^2} &= \frac{\partial^2}{\partial x^2}\sum \varepsilon^n A_n e^{\frac{i\Theta}{\varepsilon}} \\ &= \sum \varepsilon^n \left[\epsilon^2\frac{\partial^2 A_n}{\partial X^2}e^{\frac{i\Theta}{\varepsilon}} + 2\epsilon\frac{\partial A_n}{\partial X}ie^{\frac{i\Theta}{\varepsilon}}\frac{\partial \Theta}{\partial X} + A_n\left\{-e^{\frac{i\Theta}{\epsilon}}\left(\frac{\partial \Theta}{\partial X}\right)^2 + ie^{\frac{i\Theta}{\epsilon}}\epsilon\frac{\partial^2 \Theta}{\partial X^2}\right\}\right] \\ &= \sum \varepsilon^n \left[\epsilon^2\frac{\partial^2 A_n}{\partial X^2} + 2i\epsilon\frac{\partial A_n}{\partial X}\frac{\partial \Theta}{\partial X} - \left(\frac{\partial \Theta}{\partial X}\right)^2 A_n + i\epsilon\frac{\partial^2 \Theta}{\partial X^2}A_n\right]e^{\frac{i\Theta}{\varepsilon}},\end{aligned}$$

$$\begin{aligned}\frac{\partial^2 \psi}{\partial y^2} &= \frac{\partial^2}{\partial y^2}\sum \varepsilon^n A_n e^{\frac{i\Theta}{\varepsilon}} \\ &= \sum \varepsilon^n \left[\epsilon^2\frac{\partial^2 A_n}{\partial Y^2}e^{\frac{i\Theta}{\varepsilon}} + 2\epsilon\frac{\partial A_n}{\partial Y}ie^{\frac{i\Theta}{\varepsilon}}\frac{\partial \Theta}{\partial Y} + A_n\left\{-e^{\frac{i\Theta}{\epsilon}}\left(\frac{\partial \Theta}{\partial Y}\right)^2 + ie^{\frac{i\Theta}{\epsilon}}\epsilon\frac{\partial^2 \Theta}{\partial Y^2}\right\}\right] \\ &= \sum \varepsilon^n \left[\epsilon^2\frac{\partial^2 A_n}{\partial Y^2} + 2i\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial \Theta}{\partial Y} - \left(\frac{\partial \Theta}{\partial Y}\right)^2 A_n + i\epsilon\frac{\partial^2 \Theta}{\partial Y^2}A_n\right]e^{\frac{i\Theta}{\varepsilon}},\end{aligned}$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi \\
&=\sum\varepsilon^n\left[\frac{\partial}{\partial t}\left\{\epsilon^2\frac{\partial^2 A_n}{\partial X^2}+2i\epsilon\frac{\partial A_n}{\partial X}\frac{\partial\Theta}{\partial X}-\left(\frac{\partial\Theta}{\partial X}\right)^2A_n+i\epsilon\frac{\partial^2\Theta}{\partial X^2}A_n\right.\right. \\
&\qquad\left.+\epsilon^2\frac{\partial^2 A_n}{\partial Y^2}+2i\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial\Theta}{\partial Y}-\left(\frac{\partial\Theta}{\partial Y}\right)^2A_n+i\epsilon\frac{\partial^2\Theta}{\partial Y^2}A_n\right\}e^{\frac{i\Theta}{\varepsilon}} \\
&\quad+\left\{\epsilon^2\frac{\partial^2 A_n}{\partial X^2}+2i\epsilon\frac{\partial A_n}{\partial X}\frac{\partial\Theta}{\partial X}-\left(\frac{\partial\Theta}{\partial X}\right)^2A_n+i\epsilon\frac{\partial^2\Theta}{\partial X^2}A_n\right. \\
&\qquad\left.\left.+\epsilon^2\frac{\partial^2 A_n}{\partial Y^2}e^{\frac{i\Theta}{\varepsilon}}+2i\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial\Theta}{\partial Y}-\left(\frac{\partial\Theta}{\partial Y}\right)^2A_n+i\epsilon\frac{\partial^2\Theta}{\partial Y^2}A_n\right\}\frac{\partial}{\partial t}e^{\frac{i\Theta}{\varepsilon}}\right] \\
&=\sum\varepsilon^n\left[\left\{\epsilon^2\frac{\partial^2}{\partial X^2}\frac{\partial A_n}{\partial t}+2i\epsilon\frac{\partial}{\partial X}\frac{\partial A_n}{\partial t}\frac{\partial\Theta}{\partial X}+2i\epsilon\frac{\partial A_n}{\partial X}\frac{\partial}{\partial X}\frac{\partial\Theta}{\partial t}\right.\right. \\
&\quad-\frac{\partial}{\partial t}\left(\frac{\partial\Theta}{\partial X}\right)^2A_n-\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial A_n}{\partial t}+i\epsilon\frac{\partial^2}{\partial X^2}\frac{\partial\Theta}{\partial t}A_n+i\epsilon\frac{\partial^2\Theta}{\partial X^2}\frac{\partial A_n}{\partial t} \\
&\quad+\epsilon^2\frac{\partial^2}{\partial Y^2}\frac{\partial A_n}{\partial t}+2i\epsilon\frac{\partial}{\partial Y}\frac{\partial A_n}{\partial t}\frac{\partial\Theta}{\partial Y}+2i\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial\Theta}{\partial t} \\
&\quad\left.-\frac{\partial}{\partial t}\left(\frac{\partial\Theta}{\partial Y}\right)^2A_n-\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial A_n}{\partial t}+i\epsilon\frac{\partial^2}{\partial Y^2}\frac{\partial\Theta}{\partial t}A_n+i\epsilon\frac{\partial^2\Theta}{\partial Y^2}\frac{\partial A_n}{\partial t}\right\}e^{\frac{i\Theta}{\varepsilon}} \\
&\quad+\left\{\epsilon^2\frac{\partial^2 A_n}{\partial X^2}+2i\epsilon\frac{\partial A_n}{\partial X}\frac{\partial\Theta}{\partial X}-\left(\frac{\partial\Theta}{\partial X}\right)^2A_n+i\epsilon\frac{\partial^2\Theta}{\partial X^2}A_n\right. \\
&\qquad\left.\left.+\epsilon^2\frac{\partial^2 A_n}{\partial Y^2}e^{\frac{i\Theta}{\varepsilon}}+2i\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial\Theta}{\partial Y}-\left(\frac{\partial\Theta}{\partial Y}\right)^2A_n+i\epsilon\frac{\partial^2\Theta}{\partial Y^2}A_n\right\}ie^{\frac{i\Theta}{\epsilon}}\frac{\partial\Theta}{\partial T}\right] \\
&=\sum\varepsilon^n\left[\epsilon^2\frac{\partial^2}{\partial X^2}\left(\epsilon\frac{\partial A_n}{\partial T}\right)+2i\epsilon\frac{\partial}{\partial X}\left(\epsilon\frac{\partial A_n}{\partial T}\right)\frac{\partial\Theta}{\partial X}+2i\epsilon\frac{\partial A_n}{\partial X}\frac{\partial}{\partial X}\left(\epsilon\frac{\partial\Theta}{\partial T}\right)\right. \\
&\quad-2\frac{\partial\Theta}{\partial X}\frac{\partial}{\partial X}\left(\epsilon\frac{\partial\Theta}{\partial T}\right)A_n-\left(\frac{\partial\Theta}{\partial X}\right)^2\left(\epsilon\frac{\partial A_n}{\partial T}\right)+i\epsilon\frac{\partial^2}{\partial X^2}\left(\epsilon\frac{\partial\Theta}{\partial T}\right)A_n+i\epsilon\frac{\partial^2\Theta}{\partial X^2}\left(\epsilon\frac{\partial A_n}{\partial T}\right) \\
&\quad+\epsilon^2\frac{\partial^2}{\partial Y^2}\left(\epsilon\frac{\partial A_n}{\partial T}\right)+2i\epsilon\frac{\partial}{\partial Y}\left(\epsilon\frac{\partial A_n}{\partial T}\right)\frac{\partial\Theta}{\partial Y}+2i\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial}{\partial Y}\left(\epsilon\frac{\partial\Theta}{\partial T}\right) \\
&\quad-2\frac{\partial\Theta}{\partial Y}\frac{\partial}{\partial Y}\left(\epsilon\frac{\partial\Theta}{\partial T}\right)A_n-\left(\frac{\partial\Theta}{\partial Y}\right)^2\epsilon\frac{\partial A_n}{\partial T}+i\epsilon\frac{\partial^2}{\partial Y^2}\left(\epsilon\frac{\partial\Theta}{\partial T}\right)A_n+i\epsilon\frac{\partial^2\Theta}{\partial Y^2}\epsilon\frac{\partial A_n}{\partial T} \\
&\quad+\left\{i\epsilon^2\frac{\partial^2 A_n}{\partial X^2}\frac{\partial\Theta}{\partial T}-2\epsilon\frac{\partial A_n}{\partial X}\frac{\partial\Theta}{\partial X}\frac{\partial\Theta}{\partial T}-i\left(\frac{\partial\Theta}{\partial X}\right)^2A_n\frac{\partial\Theta}{\partial T}-\epsilon\frac{\partial^2\Theta}{\partial X^2}A_n\frac{\partial\Theta}{\partial T}\right. \\
&\qquad\left.\left.+i\epsilon^2\frac{\partial^2 A_n}{\partial Y^2}\frac{\partial\Theta}{\partial T}-2\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial\Theta}{\partial Y}\frac{\partial\Theta}{\partial T}-i\left(\frac{\partial\Theta}{\partial Y}\right)^2A_n\frac{\partial\Theta}{\partial T}-\epsilon\frac{\partial^2\Theta}{\partial Y^2}A_n\frac{\partial\Theta}{\partial T}\right\}\right]e^{\frac{i\Theta}{\varepsilon}},
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi \\
&=\sum\varepsilon^n\left[\epsilon^3\frac{\partial^2}{\partial X^2}\frac{\partial A_n}{\partial T}+2i\epsilon^2\frac{\partial}{\partial X}\frac{\partial A_n}{\partial T}\frac{\partial\Theta}{\partial X}+2i\epsilon^2\frac{\partial A_n}{\partial X}\frac{\partial}{\partial X}\frac{\partial\Theta}{\partial T}\right.\\
&\quad-2\epsilon\frac{\partial\Theta}{\partial X}\frac{\partial}{\partial X}\frac{\partial\Theta}{\partial T}A_n-\epsilon\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial A_n}{\partial T}+i\epsilon^2\frac{\partial^2}{\partial X^2}\frac{\partial\Theta}{\partial T}A_n+i\epsilon^2\frac{\partial^2\Theta}{\partial X^2}\frac{\partial A_n}{\partial T}\\
&\quad+\epsilon^3\frac{\partial^2}{\partial Y^2}\frac{\partial A_n}{\partial T}+2i\epsilon^2\frac{\partial}{\partial Y}\frac{\partial A_n}{\partial T}\frac{\partial\Theta}{\partial Y}+2i\epsilon^2\frac{\partial A_n}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial\Theta}{\partial T}\\
&\quad-2\epsilon\frac{\partial\Theta}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial\Theta}{\partial T}A_n-\epsilon\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial A_n}{\partial T}+i\epsilon^2\frac{\partial^2}{\partial Y^2}\frac{\partial\Theta}{\partial T}A_n+i\epsilon^2\frac{\partial^2\Theta}{\partial Y^2}\frac{\partial A_n}{\partial T}\\
&\quad+i\epsilon^2\frac{\partial^2 A_n}{\partial X^2}\frac{\partial\Theta}{\partial T}-2\epsilon\frac{\partial A_n}{\partial X}\frac{\partial\Theta}{\partial X}\frac{\partial\Theta}{\partial T}-i\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial\Theta}{\partial T}A_n-\epsilon\frac{\partial^2\Theta}{\partial X^2}\frac{\partial\Theta}{\partial T}A_n\\
&\qquad\left.+i\epsilon^2\frac{\partial^2 A_n}{\partial Y^2}\frac{\partial\Theta}{\partial T}-2\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial\Theta}{\partial Y}\frac{\partial\Theta}{\partial T}-i\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial\Theta}{\partial T}A_n-\epsilon\frac{\partial^2\Theta}{\partial Y^2}\frac{\partial\Theta}{\partial T}A_n\right]e^{\frac{i\Theta}{\varepsilon}}\\
&=\sum\varepsilon^n\left[-2\epsilon\frac{\partial\Theta}{\partial X}\frac{\partial}{\partial X}\frac{\partial\Theta}{\partial T}A_n-\epsilon\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial A_n}{\partial T}\right.\\
&\quad-2\epsilon\frac{\partial\Theta}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial\Theta}{\partial T}A_n-\epsilon\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial A_n}{\partial T}\\
&\quad-2\epsilon\frac{\partial A_n}{\partial X}\frac{\partial\Theta}{\partial X}\frac{\partial\Theta}{\partial T}-i\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial\Theta}{\partial T}A_n-\epsilon\frac{\partial^2\Theta}{\partial X^2}\frac{\partial\Theta}{\partial T}A_n\\
&\qquad\left.-2\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial\Theta}{\partial Y}\frac{\partial\Theta}{\partial T}-i\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial\Theta}{\partial T}A_n-\epsilon\frac{\partial^2\Theta}{\partial Y^2}\frac{\partial\Theta}{\partial T}A_n+O(\epsilon^2)\right]e^{\frac{i\Theta}{\varepsilon}}\\
&=\sum\varepsilon^n\left[-i\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial\Theta}{\partial T}A_n-i\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial\Theta}{\partial T}A_n\right.\\
&\quad-2\epsilon\frac{\partial\Theta}{\partial X}\frac{\partial}{\partial X}\frac{\partial\Theta}{\partial T}A_n-2\epsilon\frac{\partial\Theta}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial\Theta}{\partial T}A_n-\epsilon\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial A_n}{\partial T}-\epsilon\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial A_n}{\partial T}\\
&\quad-2\epsilon\frac{\partial A_n}{\partial X}\frac{\partial\Theta}{\partial X}\frac{\partial\Theta}{\partial T}-2\epsilon\frac{\partial A_n}{\partial Y}\frac{\partial\Theta}{\partial Y}\frac{\partial\Theta}{\partial T}-\epsilon\frac{\partial^2\Theta}{\partial X^2}\frac{\partial\Theta}{\partial T}A_n-\epsilon\frac{\partial^2\Theta}{\partial Y^2}\frac{\partial\Theta}{\partial T}A_n\\
&\qquad\left.+O(\epsilon^2)\right]e^{\frac{i\Theta}{\varepsilon}}.
\end{aligned}
$$

以上を渦度方程式 (2.11)

$$
\frac{\partial}{\partial t}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\psi+\beta\frac{\partial\psi}{\partial x}=0
$$

に代入すると,

$$
\begin{aligned}
&\sum \varepsilon^n \left[-i \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial \Theta}{\partial T} A_n - i \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial \Theta}{\partial T} A_n \right. \\
&\quad - 2\epsilon \frac{\partial \Theta}{\partial X}\frac{\partial}{\partial X}\frac{\partial \Theta}{\partial T} A_n - 2\epsilon \frac{\partial \Theta}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial \Theta}{\partial T} A_n - \epsilon \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial A_n}{\partial T} - \epsilon \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial A_n}{\partial T} \\
&\quad - 2\epsilon \frac{\partial A_n}{\partial X}\frac{\partial \Theta}{\partial X}\frac{\partial \Theta}{\partial T} - 2\epsilon \frac{\partial A_n}{\partial Y}\frac{\partial \Theta}{\partial Y}\frac{\partial \Theta}{\partial T} - \epsilon \frac{\partial^2 \Theta}{\partial X^2}\frac{\partial \Theta}{\partial T} A_n - \epsilon \frac{\partial^2 \Theta}{\partial Y^2}\frac{\partial \Theta}{\partial T} A_n \\
&\qquad \left. + O(\epsilon^2) \right] e^{\frac{i\Theta}{\varepsilon}} \\
&+ \beta \sum \varepsilon^n \left[\epsilon \frac{\partial A_n}{\partial X} + i \frac{\partial \Theta}{\partial X} A_n \right] e^{\frac{i\Theta}{\varepsilon}} = 0.
\end{aligned}
$$

各オーダの式を取りだすと以下のようになる.

- $O(\epsilon^0)$ の式

$$
\left[-i \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial \Theta}{\partial T} A_0 - i \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial \Theta}{\partial T} A_0 + i\beta \frac{\partial \Theta}{\partial X} A_0 \right] e^{\frac{i\Theta}{\varepsilon}} = 0,
$$

$$
\left[-i \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial \Theta}{\partial T} - i \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial \Theta}{\partial T} + i\beta \frac{\partial \Theta}{\partial X} \right] A_0 = 0. \tag{2.34}
$$

これで (2.21) が得られた.

- $O(\epsilon^1)$ の式

$$
\begin{aligned}
&- i\epsilon \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial \Theta}{\partial T} A_1 - i\epsilon \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial \Theta}{\partial T} A_1 \\
&\quad - 2\epsilon \frac{\partial \Theta}{\partial X}\frac{\partial}{\partial X}\frac{\partial \Theta}{\partial T} A_0 - 2\epsilon \frac{\partial \Theta}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial \Theta}{\partial T} A_0 - \epsilon \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial A_0}{\partial T} - \epsilon \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial A_0}{\partial T} \\
&\quad - 2\epsilon \frac{\partial A_0}{\partial X}\frac{\partial \Theta}{\partial X}\frac{\partial \Theta}{\partial T} - 2\epsilon \frac{\partial A_0}{\partial Y}\frac{\partial \Theta}{\partial Y}\frac{\partial \Theta}{\partial T} - \epsilon \frac{\partial^2 \Theta}{\partial X^2}\frac{\partial \Theta}{\partial T} A_0 - \epsilon \frac{\partial^2 \Theta}{\partial Y^2}\frac{\partial \Theta}{\partial T} A_0 \\
&\quad + \epsilon\beta \frac{\partial A_0}{\partial X} + i\epsilon\beta \frac{\partial \Theta}{\partial X} A_1 = 0, \\
&\left[-i \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial \Theta}{\partial T} - i \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial \Theta}{\partial T} + i\beta \frac{\partial \Theta}{\partial X} \right] A_1 \\
&\quad + \left[-2 \frac{\partial \Theta}{\partial X}\frac{\partial}{\partial X}\frac{\partial \Theta}{\partial T} - 2 \frac{\partial \Theta}{\partial Y}\frac{\partial}{\partial Y}\frac{\partial \Theta}{\partial T} - \frac{\partial^2 \Theta}{\partial X^2}\frac{\partial \Theta}{\partial T} - \frac{\partial^2 \Theta}{\partial Y^2}\frac{\partial \Theta}{\partial T} \right] A_0 \\
&\quad + \left[- \left(\frac{\partial \Theta}{\partial X}\right)^2 \frac{\partial A_0}{\partial T} - \left(\frac{\partial \Theta}{\partial Y}\right)^2 \frac{\partial A_0}{\partial T} - 2 \frac{\partial \Theta}{\partial X}\frac{\partial \Theta}{\partial T}\frac{\partial A_0}{\partial X} - 2 \frac{\partial \Theta}{\partial Y}\frac{\partial \Theta}{\partial T}\frac{\partial A_0}{\partial Y} + \beta \frac{\partial A_0}{\partial X} \right] = 0.
\end{aligned}
$$

これで (2.22) が得られた. 分散関係を使うと

$$\left[-2\frac{\partial}{\partial X}\frac{\partial\Theta}{\partial T}\frac{\partial\Theta}{\partial X}-2\frac{\partial}{\partial Y}\frac{\partial\Theta}{\partial T}\frac{\partial\Theta}{\partial Y}-\frac{\partial^2\Theta}{\partial X^2}\frac{\partial\Theta}{\partial T}-\frac{\partial^2\Theta}{\partial Y^2}\frac{\partial\Theta}{\partial T}\right]A_0$$
$$-\left(\frac{\partial\Theta}{\partial X}\right)^2\frac{\partial A_0}{\partial T}-\left(\frac{\partial\Theta}{\partial Y}\right)^2\frac{\partial A_0}{\partial T}-2\frac{\partial\Theta}{\partial X}\frac{\partial\Theta}{\partial T}\frac{\partial A_0}{\partial X}-2\frac{\partial\Theta}{\partial Y}\frac{\partial\Theta}{\partial T}\frac{\partial A_0}{\partial Y}+\beta\frac{\partial A_0}{\partial X}=0$$

となる.

局所振動数, 局所波数

$$(k,l)\equiv\left(\frac{\partial\Theta}{\partial X},\frac{\partial\Theta}{\partial Y}\right)$$
$$\omega\equiv-\frac{\partial\Theta}{\partial T}$$

を使うと $O(\epsilon^0)$ の式は

$$-k^2(-\omega)-l^2(-\omega)+\beta k=0,$$
$$(k^2+l^2)\omega+\beta k=0$$

となる.

$O(\epsilon^1)$ の式は

$$\left[-2\frac{\partial k}{\partial T}k-2\frac{\partial l}{\partial T}l-\frac{\partial k}{\partial X}(-\omega)-\frac{\partial k}{\partial Y}(-\omega)\right]A_0$$
$$-k^2\frac{\partial A_0}{\partial T}-l^2\frac{\partial A_0}{\partial T}-2k(-\omega)\frac{\partial A_0}{\partial X}-2l(-\omega)\frac{\partial A_0}{\partial Y}+\beta\frac{\partial A_0}{\partial X}=0,$$
$$\left[-2\frac{\partial k}{\partial T}k-2\frac{\partial l}{\partial T}l+\frac{\partial k}{\partial X}\omega+\frac{\partial k}{\partial Y}\omega\right]A_0$$
$$-(k^2+l^2)\frac{\partial A_0}{\partial T}+2k\omega\frac{\partial A_0}{\partial X}+2l\omega\frac{\partial A_0}{\partial Y}+\beta\frac{\partial A_0}{\partial X}=0.$$

これで (2.26) が得られた.

2.6.3 エネルギーの保存則

(2.26) を書き換えて保存形に変換する. (2.26)× A_0^* + (2.26)* × A_0 を作ると

$$\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\}+\frac{\partial}{\partial X}\left\{(-2\omega k-\beta)|A_0|^2\right\}+\frac{\partial}{\partial Y}\left\{(-2\omega l)|A_0|^2\right\}=0 \quad (2.35)$$

群速度を用いて書き換えると

$$\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\}+\frac{\partial}{\partial X}\left\{c_{gx}(k^2+l^2)|A_0|^2\right\}+\frac{\partial}{\partial Y}\left\{c_{gy}(k^2+l^2)|A_0|^2\right\}=0 \tag{2.36}$$

この式は $(k^2+l^2)|A_0|^2$ という量が保存し, そのフラックスが $(\boldsymbol{c}_g k^2+l^2)|A_0|^2$ で表されることを示す.

この量は運動エネルギーにほかならない. ε^0 次のオーダーでは

$$u=-ilA_0e^{i\Theta/\varepsilon}, \tag{2.37}$$

$$v=ikA_0e^{i\Theta/\varepsilon} \tag{2.38}$$

であるから運動エネルギー $E\equiv\rho(u^2+v^2)/2$ の Θ 平均は

$$\begin{aligned}\bar{E}&=\frac{\rho}{2}(\overline{u^2}+\overline{v^2}) \\ &=\frac{\rho}{4}(k^2+l^2)|A_0|^2.\end{aligned} \tag{2.39}$$

密度 ρ のファクターを除けば全く同じ量であることが示された.

2.6.4　付録: エネルギーの保存則の導出

ここでは, エネルギー保存則 (2.36) を導出する.

(2.26) を書き換えて保存形に変換する. (2.26) $\times$ A_0^* + (2.26)* $\times$ A_0 を作ると

$$\begin{aligned}
&\left[\left\{-2\frac{\partial k}{\partial T}k+\frac{\partial k}{\partial X}\omega-2\frac{\partial l}{\partial T}l+\frac{\partial l}{\partial Y}\omega\right\}A_0-(k^2+l^2)\frac{\partial A_0}{\partial T}+2k\omega\frac{\partial A_0}{\partial X}+2l\omega\frac{\partial A_0}{\partial Y}+\beta\frac{\partial A_0}{\partial X}\right]A_0^* \\
&\quad+\left[\left\{-2\frac{\partial k}{\partial T}k+\frac{\partial k}{\partial X}\omega-2\frac{\partial l}{\partial T}l+\frac{\partial l}{\partial Y}\omega\right\}A_0^*-(k^2+l^2)\frac{\partial A_0^*}{\partial T}+2k\omega\frac{\partial A_0^*}{\partial X}+2l\omega\frac{\partial A_0^*}{\partial Y}+\beta\frac{\partial A_0^*}{\partial X}\right]A_0=0, \\
&2\left\{-\frac{\partial k^2}{\partial T}+\frac{\partial k}{\partial X}\omega-\frac{\partial l^2}{\partial T}+\frac{\partial l}{\partial Y}\omega\right\}|A_0|^2-(k^2+l^2)\frac{\partial|A_0|^2}{\partial T}+2k\omega\frac{\partial|A_0|^2}{\partial X}+2l\omega\frac{\partial|A_0|^2}{\partial Y}+\beta\frac{\partial|A_0|^2}{\partial X}=0, \\
&-2\frac{\partial k^2}{\partial T}|A_0|^2+2\frac{\partial k}{\partial X}\omega|A_0|^2-2\frac{\partial l^2}{\partial T}|A_0|^2+2\frac{\partial l}{\partial Y}\omega|A_0|^2-(k^2+l^2)\frac{\partial|A_0|^2}{\partial T} \\
&\quad+2k\omega\frac{\partial|A_0|^2}{\partial X}+2l\omega\frac{\partial|A_0|^2}{\partial Y}+\beta\frac{\partial|A_0|^2}{\partial X}=0, \\
&-\frac{\partial k^2}{\partial T}|A_0|^2-\frac{\partial l^2}{\partial T}|A_0|^2-(k^2+l^2)\frac{\partial|A_0|^2}{\partial T}+2\omega\frac{\partial k}{\partial X}|A_0|^2+2\omega k\frac{\partial|A_0|^2}{\partial X}+\beta\frac{\partial|A_0|^2}{\partial X} \\
&\quad+2\omega\frac{\partial l}{\partial Y}|A_0|^2+2\omega l\frac{\partial|A_0|^2}{\partial Y}-\frac{\partial k^2}{\partial T}|A_0|^2-\frac{\partial l^2}{\partial T}|A_0|^2=0, \\
&-\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\}+2\omega\frac{\partial k}{\partial X}|A_0|^2+2\omega k\frac{\partial|A_0|^2}{\partial X}+\beta\frac{\partial|A_0|^2}{\partial X} \\
&\quad+2\omega\frac{\partial l}{\partial Y}|A_0|^2+2\omega l\frac{\partial|A_0|^2}{\partial Y}-\frac{\partial k^2}{\partial T}|A_0|^2-\frac{\partial l^2}{\partial T}|A_0|^2=0.
\end{aligned}$$

ここで, 分散関係

$$\omega = -\frac{\beta k}{k^2+l^2}$$

を使うと,

$$\begin{aligned}
&-\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\} + 2\left(-\frac{\beta k}{k^2+l^2}\right)\frac{\partial k}{\partial X}|A_0|^2 + 2\left(-\frac{\beta k}{k^2+l^2}\right)k\frac{\partial |A_0|^2}{\partial X} + \beta\frac{\partial |A_0|^2}{\partial X} \\
&+ 2\left(-\frac{\beta k}{k^2+l^2}\right)\frac{\partial l}{\partial Y}|A_0|^2 + 2\left(-\frac{\beta k}{k^2+l^2}\right)l\frac{\partial |A_0|^2}{\partial Y} - \frac{\partial k^2}{\partial T}|A_0|^2 - \frac{\partial l^2}{\partial T}|A_0|^2 = 0, \\
&-\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\} + 2\left(-\frac{\beta k}{k^2+l^2}\right)\frac{\partial k}{\partial X}|A_0|^2 + \frac{-2\beta k^2 + \beta(k^2+l^2)}{k^2+l^2}\frac{\partial |A_0|^2}{\partial X} \\
&+ 2\left(-\frac{\beta k}{k^2+l^2}\right)\frac{\partial l}{\partial Y}|A_0|^2 + 2\left(-\frac{\beta k}{k^2+l^2}\right)l\frac{\partial |A_0|^2}{\partial Y} - \frac{\partial k^2}{\partial T}|A_0|^2 - \frac{\partial l^2}{\partial T}|A_0|^2 = 0, \\
&-\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\} - \frac{2\beta k}{k^2+l^2}\frac{\partial k}{\partial X}|A_0|^2 + \frac{\beta(l^2-k^2)}{k^2+l^2}\frac{\partial |A_0|^2}{\partial X} \\
&- \frac{2\beta k}{k^2+l^2}\frac{\partial l}{\partial Y}|A_0|^2 - \frac{2\beta kl}{k^2+l^2}\frac{\partial |A_0|^2}{\partial Y} - \frac{\partial k^2}{\partial T}|A_0|^2 - \frac{\partial l^2}{\partial T}|A_0|^2 = 0,
\end{aligned}$$

となる. k^2, l^2 の T 微分を書き換える.

$$\begin{aligned}
&\frac{\partial k^2}{\partial T} = 2k\frac{\partial k}{\partial T} = 2k\frac{\partial}{\partial T}\frac{\partial \Theta}{\partial X} = 2k\frac{\partial}{\partial X}\frac{\partial \Theta}{\partial T} = -2k\frac{\partial \omega}{\partial X} = -2k\frac{\partial}{\partial X}\left(-\frac{\beta k}{k^2+l^2}\right) \\
&= -2k\left(-\beta\frac{\frac{\partial k}{\partial X}(k^2+l^2) - k(2k\frac{\partial k}{\partial X} + 2l\frac{\partial l}{\partial X})}{(k^2+l^2)^2}\right) \\
&= 2k\beta\frac{\frac{\partial k}{\partial X}(k^2+l^2) - 2k^2\frac{\partial k}{\partial X} - 2kl\frac{\partial l}{\partial X}}{(k^2+l^2)^2} = 2k\beta\frac{(l^2-k^2)\frac{\partial k}{\partial X} - 2kl\frac{\partial l}{\partial X}}{(k^2+l^2)^2} \\
&= \beta\frac{2k(l^2-k^2)\frac{\partial k}{\partial X} - 4k^2l\frac{\partial l}{\partial X}}{(k^2+l^2)^2}, \\
&\frac{\partial l^2}{\partial T} = 2l\frac{\partial l}{\partial T} = 2l\frac{\partial}{\partial T}\frac{\partial \Theta}{\partial Y} = 2l\frac{\partial}{\partial Y}\frac{\partial \Theta}{\partial T} = -2l\frac{\partial \omega}{\partial Y} = -2l\frac{\partial}{\partial Y}\left(-\frac{\beta k}{k^2+l^2}\right) \\
&= -2l\left(-\beta\frac{\frac{\partial k}{\partial Y}(k^2+l^2) - k(2k\frac{\partial k}{\partial Y} + 2l\frac{\partial l}{\partial Y})}{(k^2+l^2)^2}\right) \\
&= 2l\beta\frac{\frac{\partial k}{\partial Y}(k^2+l^2) - 2k^2\frac{\partial k}{\partial Y} - 2kl\frac{\partial l}{\partial Y}}{(k^2+l^2)^2} = 2l\beta\frac{(l^2-k^2)\frac{\partial k}{\partial Y} - 2kl\frac{\partial l}{\partial Y}}{(k^2+l^2)^2} \\
&= \beta\frac{2l(l^2-k^2)\frac{\partial k}{\partial Y} - 4kl^2\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}.
\end{aligned} \tag{2.40}$$

これらを $O(\epsilon)$ の式に再び代入すると

$$
\begin{aligned}
&-\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\}-\frac{2\beta k}{k^2+l^2}\frac{\partial k}{\partial X}|A_0|^2+\frac{\beta(l^2-k^2)}{k^2+l^2}\frac{\partial |A_0|^2}{\partial X}\\
&\quad-\frac{2\beta k}{k^2+l^2}\frac{\partial l}{\partial Y}|A_0|^2-\frac{2\beta kl}{k^2+l^2}\frac{\partial |A_0|^2}{\partial Y}\\
&\quad-\beta\frac{2k(l^2-k^2)\frac{\partial k}{\partial X}-4k^2l\frac{\partial l}{\partial X}}{(k^2+l^2)^2}|A_0|^2-\beta\frac{2l(l^2-k^2)\frac{\partial k}{\partial Y}-4kl^2\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2=0,\\
&\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\}+\beta\frac{2k}{k^2+l^2}\frac{\partial k}{\partial X}|A_0|^2+\beta\frac{2k}{k^2+l^2}\frac{\partial l}{\partial Y}|A_0|^2\\
&\quad+\beta\frac{2k(l^2-k^2)\frac{\partial k}{\partial X}-4k^2l\frac{\partial l}{\partial X}}{(k^2+l^2)^2}|A_0|^2+\beta\frac{2l(l^2-k^2)\frac{\partial k}{\partial Y}-4kl^2\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2\\
&\quad+\frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial |A_0|^2}{\partial X}+\frac{2\beta kl}{k^2+l^2}\frac{\partial |A_0|^2}{\partial Y}=0,\\
&\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\}\\
&\quad+\beta\frac{2k(k^2+l^2)\frac{\partial k}{\partial X}+2k(k^2+l^2)\frac{\partial l}{\partial Y}+2k(l^2-k^2)\frac{\partial k}{\partial X}-4k^2l\frac{\partial l}{\partial X}+2l(l^2-k^2)\frac{\partial k}{\partial Y}-4kl^2\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2\\
&\quad+\frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial |A_0|^2}{\partial X}+\frac{2\beta kl}{k^2+l^2}\frac{\partial |A_0|^2}{\partial Y}=0,\\
&\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\}+\beta\frac{4kl^2\frac{\partial k}{\partial X}-4k^2l\frac{\partial l}{\partial X}+2l(l^2-k^2)\frac{\partial k}{\partial Y}+2k(k^2-l^2)\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2\\
&\quad+\frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial |A_0|^2}{\partial X}+\frac{2\beta kl}{k^2+l^2}\frac{\partial |A_0|^2}{\partial Y}=0.
\end{aligned}
$$

群速度

$$
\begin{aligned}
c_{gx}(k^2+l^2)&=\frac{\beta(k^2-l^2)}{k^2+l^2},\\
c_{gy}(k^2+l^2)&=\frac{2\beta kl}{k^2+l^2},
\end{aligned}
$$

を使って書き換える.

$$\frac{\partial}{\partial X}(c_{gx}(k^2+l^2)|A_0|^2) = \frac{\partial}{\partial X}\left(\frac{\beta(k^2-l^2)}{k^2+l^2}|A_0|^2\right)$$
$$= \frac{\partial}{\partial X}\left(\frac{\beta(k^2-l^2)}{k^2+l^2}\right)|A_0|^2 + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X}$$
$$= \frac{\beta(2k\frac{\partial k}{\partial X}-2l\frac{\partial l}{\partial X})(k^2+l^2)-\beta(k^2-l^2)(2k\frac{\partial k}{\partial X}+2l\frac{\partial l}{\partial X})}{(k^2+l^2)^2}|A_0|^2 + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X}$$
$$= \frac{\beta(2k(k^2+l^2)\frac{\partial k}{\partial X}-2l(k^2+l^2)\frac{\partial l}{\partial X}-(k^2-l^2)(2k\frac{\partial k}{\partial X}+2l\frac{\partial l}{\partial X})}{(k^2+l^2)^2}|A_0|^2 + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X}$$
$$= \beta\frac{2k(k^2+l^2)\frac{\partial k}{\partial X}-2l(k^2+l^2)\frac{\partial l}{\partial X}-2k(k^2-l^2)\frac{\partial k}{\partial X}-2l(k^2-l^2)\frac{\partial l}{\partial X}}{(k^2+l^2)^2}|A_0|^2 + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X}$$
$$= \beta\frac{2k(2l^2)\frac{\partial k}{\partial X}-2l(2k^2)\frac{\partial l}{\partial X}}{(k^2+l^2)^2}|A_0|^2 + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X}$$
$$= \beta\frac{4kl^2\frac{\partial k}{\partial X}-4k^2l\frac{\partial l}{\partial X}}{(k^2+l^2)^2}|A_0|^2 + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X},$$
$$\frac{\partial}{\partial Y}(c_{gy}(k^2+l^2)|A_0|^2) = \frac{\partial}{\partial Y}\left(\frac{2\beta kl}{k^2+l^2}|A_0|^2\right)$$
$$= \frac{\partial}{\partial Y}\left(\frac{2\beta kl}{k^2+l^2}\right)|A_0|^2 + \left(\frac{2\beta kl}{k^2+l^2}\right)\frac{\partial|A_0|^2}{\partial Y}$$
$$= \beta\frac{(2\frac{\partial k}{\partial Y}l+2k\frac{\partial l}{\partial Y})(k^2+l^2)-(2kl)(2k\frac{\partial k}{\partial Y}+2l\frac{\partial l}{\partial Y})}{(k^2+l^2)^2}|A_0|^2 + \left(\frac{2\beta kl}{k^2+l^2}\right)\frac{\partial|A_0|^2}{\partial Y}$$
$$= \beta\frac{2l(k^2+l^2)\frac{\partial k}{\partial Y}+2k(k^2+l^2)\frac{\partial l}{\partial Y}-4k^2l\frac{\partial k}{\partial Y}-4kl^2\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2 + \left(\frac{2\beta kl}{k^2+l^2}\right)\frac{\partial|A_0|^2}{\partial Y}$$
$$= \beta\frac{2l(l^2-k^2)\frac{\partial k}{\partial Y}+2k(k^2-l^2)\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2 + \left(\frac{2\beta kl}{k^2+l^2}\right)\frac{\partial|A_0|^2}{\partial Y}$$

これから,

$$\begin{aligned}
&\frac{\partial}{\partial X}(c_{gx}(k^2+l^2)|A_0|^2) + \frac{\partial}{\partial Y}(c_{gy}(k^2+l^2)|A_0|^2) \\
&= \beta\frac{4kl^2\frac{\partial k}{\partial X}-4k^2l\frac{\partial l}{\partial X}}{(k^2+l^2)^2}|A_0|^2 + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X} \\
&\quad + \beta\frac{2l(l^2-k^2)\frac{\partial k}{\partial Y}+2k(k^2-l^2)\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2 + \left(\frac{2\beta kl}{k^2+l^2}\right)\frac{\partial|A_0|^2}{\partial Y} \\
&= \beta\frac{4kl^2\frac{\partial k}{\partial X}-4k^2l\frac{\partial l}{\partial X}+2l(l^2-k^2)\frac{\partial k}{\partial Y}+2k(k^2-l^2)\frac{\partial l}{\partial Y}}{(k^2+l^2)^2}|A_0|^2 \\
&\quad + \frac{\beta(k^2-l^2)}{k^2+l^2}\frac{\partial|A_0|^2}{\partial X} + \left(\frac{2\beta kl}{k^2+l^2}\right)\frac{\partial|A_0|^2}{\partial Y}
\end{aligned}$$

よって, 以下の式が得られる.

$$\frac{\partial}{\partial T}\left\{(k^2+l^2)|A_0|^2\right\} + \frac{\partial}{\partial X}\left\{c_{gx}(k^2+l^2)|A_0|^2\right\} + \frac{\partial}{\partial Y}\left\{c_{gy}(k^2+l^2)|A_0|^2\right\} = 0$$

これで (2.36) が得られた.

2.6.5 エネルギーの保存則, 直接的な導出

線型化された渦度方程式 (2.11) から直接出発してエネルギーの式を導いてみる. (2.11) に ψ をかけて変形すると

$$\frac{\partial}{\partial t}\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]+\frac{\partial}{\partial x}\left[-\psi\frac{\partial^2\psi}{\partial t\partial x}-\frac{1}{2}\beta\psi^2\right]+\frac{\partial}{\partial y}\left[-\psi\frac{\partial^2\psi}{\partial t\partial y}\right]=0. \tag{2.41}$$

ψ を (2.18) 型の解とすれば, 位相平均することにより $O(\varepsilon^0)$ では

$$\overline{\frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2+\left(\frac{\partial\psi}{\partial y}\right)^2\right]}=\frac{1}{4}(k^2+l^2)|A_0|^2\ =\ \bar{E}, \tag{2.42}$$

$$\overline{-\psi\frac{\partial^2\psi}{\partial t\partial x}-\frac{1}{2}\beta\psi^2}=-\left(\frac{1}{2}k\omega+\frac{1}{4}\beta\right)|A_0|^2\ =\ c_{gx}\bar{E}, \tag{2.43}$$

$$\overline{-\psi\frac{\partial^2\psi}{\partial t\partial y}}=-\frac{1}{2}l\omega|A_0|^2\ =\ c_{gy}\bar{E} \tag{2.44}$$

である.

2.7 エネルギーの保存則に関する注意

線型化された運動方程式 (2.8) ∼ (2.10) から直接出発してエネルギーの式を導いてみる. (2.9) × u + (2.10) × v を計算した後 (2.8) を使って整理すれば

$$\frac{\partial}{\partial t}\left(\frac{\rho}{2}\boldsymbol{v}^2\right)+\nabla\cdot(p\boldsymbol{v})=0.$$

が得られる. ここで, 注意するべきことは $\nabla\cdot\overline{p\boldsymbol{v}}=\nabla\cdot(p\boldsymbol{v})$ は成り立つけれども, エネルギーフラックスの位相平均については

$$\overline{p\boldsymbol{v}}\neq\boldsymbol{c}_g\bar{E} \tag{2.45}$$

となることである. 以下では, このことを確認する.

$\overline{p\boldsymbol{v}}$ の評価は次のように行なう. (2.9) から圧力擾乱をもとめればよいのであるが, f はゆっくりとは変わらない (その微分が β である) 変数であることに注意しなければならない:

$$\beta=\frac{\partial f}{\partial y}\ =\ \frac{1}{\varepsilon}\frac{\partial f}{\partial Y}.$$

従って, p の振幅にもゆっくりとは変化しない成分が存在することになる. p は ε で次のように展開する.

$$p = \sum_n \varepsilon^n p_n e^{i\Theta/\varepsilon}. \tag{2.46}$$

ただし p_n は y に関しては必ずしもゆっくりとは変化しない関数である. (2.9) から p をもとめる. $\boldsymbol{v}$ を ε^1 次まで計算しておくと

$$\begin{aligned} u &= -\frac{\partial\psi}{\partial y} = -\frac{\partial}{\partial y}\left(\sum_{n=0}\varepsilon^n A_n(\boldsymbol{X},T)e^{i\Theta(\boldsymbol{x},t)/\varepsilon}\right) = -\sum_{n=0}\varepsilon^n\left[\epsilon\frac{\partial A_n}{\partial Y} + iA_n\frac{\partial\Theta}{\partial Y}\right]e^{i\Theta(\boldsymbol{x},t)/\varepsilon} \\ &= \left(-ilA_0 - \varepsilon\frac{\partial A_0}{\partial Y} - \varepsilon ilA_1\right)e^{i\Theta/\varepsilon} \end{aligned} \tag{2.47}$$

$$v = \frac{\partial\psi}{\partial x} = \left(ikA_0 + \varepsilon\frac{\partial A_0}{\partial X} + \varepsilon ikA_1\right)e^{i\Theta/\varepsilon} \tag{2.48}$$

更に,

$$\begin{aligned} \frac{\partial u}{\partial t} &= \frac{\partial}{\partial t}\left[\left(-ilA_0 - \varepsilon\frac{\partial A_0}{\partial Y} - \varepsilon ilA_1\right)e^{i\Theta/\varepsilon}\right] \\ &= \left(-il\varepsilon\frac{\partial A_0}{\partial T} - \varepsilon^2\frac{\partial}{\partial T}\frac{\partial A_0}{\partial Y} - \varepsilon^2 il\frac{\partial A_1}{\partial T}\right)e^{i\Theta/\varepsilon} + \left[\left(-ilA_0 - \varepsilon\frac{\partial A_0}{\partial Y} - \varepsilon ilA_1\right)ie^{i\Theta/\varepsilon}\right]\frac{\partial\Theta}{\partial T} \\ &= -il\varepsilon\frac{\partial A_0}{\partial T} - \left(-ilA_0 - \varepsilon\frac{\partial A_0}{\partial Y} - \varepsilon ilA_1\right)ie^{i\Theta/\varepsilon}\omega \\ &= -il\varepsilon\frac{\partial A_0}{\partial T} + i\omega\left(ilA_0 + \varepsilon\frac{\partial A_0}{\partial Y} + \varepsilon ilA_1\right)e^{i\Theta/\varepsilon}, \\ \frac{\partial p}{\partial x} &= \left(ikp_0 + \varepsilon\frac{\partial p_0}{\partial X} + \varepsilon ikp_1\right)e^{i\Theta/\varepsilon} \end{aligned}$$

となる. これらを (2.9) に代入する.

$$\begin{aligned} &-il\varepsilon\frac{\partial A_0}{\partial T} + i\omega\left(ilA_0 + \varepsilon\frac{\partial A_0}{\partial Y} + \varepsilon ilA_1\right)e^{i\Theta/\varepsilon} - f\left(ikA_0 + \varepsilon\frac{\partial A_0}{\partial X} + \varepsilon ikA_1\right)e^{i\Theta/\varepsilon} \\ &\quad = -\frac{1}{\rho}\left(ikp_0 + \varepsilon\frac{\partial p_0}{\partial X} + \varepsilon ikp_1\right)e^{i\Theta/\varepsilon}, \\ &-il\varepsilon\frac{\partial A_0}{\partial T} + i\omega\left(ilA_0 + \varepsilon\frac{\partial A_0}{\partial Y} + \varepsilon ilA_1\right) - f\left(ikA_0 + \varepsilon\frac{\partial A_0}{\partial X} + \varepsilon ikA_1\right) \\ &\quad = -\frac{1}{\rho}\left(ikp_0 + \varepsilon\frac{\partial p_0}{\partial X} + \varepsilon ikp_1\right), \\ &-il\varepsilon\frac{\partial A_0}{\partial T} - \omega lA_0 + i\varepsilon\omega\frac{\partial A_0}{\partial Y} - \varepsilon\omega lA_1 - ifkA_0 - \varepsilon(f_0 + \beta\varepsilon Y)\frac{\partial A_0}{\partial X} - i\varepsilon(f_0 + \beta\varepsilon Y)kA_1 \\ &\quad = -\frac{1}{\rho}\left(ikp_0 + \varepsilon\frac{\partial p_0}{\partial X} + \varepsilon ikp_1\right), \end{aligned}$$

- $O(\varepsilon^0)$ の式

$$(-l\omega - ikf_0)A_0 = -\frac{1}{\rho}ikp_0.$$

- $O(\varepsilon^1)$ の式

$$-il\frac{\partial A_0}{\partial T} + i\omega\frac{\partial A_0}{\partial Y} - \omega l A_1 - f_0\frac{\partial A_0}{\partial X} - if_0kA_1 = -\frac{1}{\rho}\frac{\partial p_0}{\partial X} - \frac{1}{\rho}ikp_1,$$
$$i\omega\frac{\partial A_0}{\partial Y} - il\frac{\partial A_0}{\partial T} - f_0\frac{\partial A_0}{\partial X} + (-l\omega - if_0k)A_1 = -\frac{1}{\rho}\frac{\partial p_0}{\partial X} - \frac{1}{\rho}ikp_1,$$

となる.

更に, p_1 について解くと,

$$ikp_1 = -\frac{\partial p_0}{\partial X} - \rho\left[i\omega\frac{\partial A_0}{\partial Y} - il\frac{\partial A_0}{\partial T} - f\frac{\partial A_0}{\partial X} + (-l\omega - ifk)A_1\right],$$
$$p_1 = -\frac{1}{ik}\frac{\partial p_0}{\partial X} - \frac{1}{ik}\rho\left[i\omega\frac{\partial A_0}{\partial Y} - il\frac{\partial A_0}{\partial T} - f\frac{\partial A_0}{\partial X} + (-l\omega - ifk)A_1\right],$$
$$p_1 = -\frac{1}{ik}\frac{\partial}{\partial X}\left\{\rho\left(f - i\omega\frac{l}{k}\right)A_0\right\} - \frac{1}{ik}\rho\left[i\omega\frac{\partial A_0}{\partial Y} - il\frac{\partial A_0}{\partial T} - f\frac{\partial A_0}{\partial X} + (-l\omega - ifk)A_1\right],$$
$$p_1 = \rho\frac{1}{k}\frac{\partial}{\partial X}\left\{\left(if + \frac{\omega l}{k}\right)A_0\right\} - \rho\left[\frac{\omega}{k}\frac{\partial A_0}{\partial Y} - \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{if}{k}\frac{\partial A_0}{\partial X} + \left(\omega\frac{il}{k} - f\right)A_1\right],$$
$$p_1 = \rho\left[\frac{1}{k}\frac{\partial}{\partial X}\left\{\left(if + \frac{\omega l}{k}\right)A_0\right\} - \frac{\omega}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} - \frac{if}{k}\frac{\partial A_0}{\partial X} - \left(\omega\frac{il}{k} - f\right)A_1\right],$$
$$p_1 = \rho\left[\frac{1}{k}\frac{\partial}{\partial X}\left(if + \frac{\omega}{k}\right)A_0 + \left(\frac{if}{k} + \frac{\omega l}{k^2}\right)\frac{\partial A_0}{\partial X} - \frac{\omega}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} - \frac{if}{k}\frac{\partial A_0}{\partial X} + \left(f - i\omega\frac{l}{k}\right)A_1\right],$$
$$p_1 = \rho\left[\frac{1}{k}\frac{\partial}{\partial X}\left(\frac{\omega l}{k}\right)A_0 + \frac{\omega l}{k^2}\frac{\partial A_0}{\partial X} - \frac{\omega}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \left(f - i\omega\frac{l}{k}\right)A_1\right].$$

を得る. よって

$$p_0 = \rho\left(f - i\omega\frac{l}{k}\right)A_0,$$
$$p_1 = \rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\}.$$

となる. 結局, ε^1 次までで

$$\overline{pu} = \frac{1}{2}\rho\omega\frac{l^2}{k}|A_0|^2 + \varepsilon\rho\left\{\frac{1}{2}\omega\frac{l^2}{k}(A_0A_1^* + A_0^*A_1) - \frac{1}{4}f\frac{\partial}{\partial Y}|A_0|^2\right\} \tag{2.49}$$
$$\overline{pv} = -\frac{1}{2}\rho\omega l|A_0|^2 + \varepsilon\rho\left\{-\frac{1}{2}\omega l(A_0A_1^* + A_0^*A_1) + \frac{1}{4}f\frac{\partial}{\partial X}|A_0|^2\right\} \tag{2.50}$$

上式の導出を以下に示す. p は

$$
\begin{aligned}
p = &\left[\rho\left(f - i\omega\frac{l}{k}\right)A_0\right.\\
&\left.+\epsilon\left\{\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\}\right\}\right]e^{i\Theta/\varepsilon}
\end{aligned}
$$

となる. pu は

$$
pu = \frac{1}{2}(p + p^*) \times \frac{1}{2}(u + u^*) = \frac{1}{4}(pu + pu^* + p^*u + p^*u^*) \tag{2.51}
$$

であり, 平均を取ると残るのは $pu^* + p^*u$ のみ. よって,

$$
\begin{aligned}
pu^* = &\left[\rho\left(f - i\omega\frac{l}{k}\right)A_0\right.\\
&\left.+\epsilon\left\{\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\}\right\}\right]e^{i\Theta/\varepsilon}\\
&\times\left(ilA_0^* - \varepsilon\frac{\partial A_0^*}{\partial Y} + \varepsilon ilA_1^*\right)e^{-i\Theta/\varepsilon}\\
= &\left[\rho\left(f - i\omega\frac{l}{k}\right)A_0 + \epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\}\right]\\
&\times\left(ilA_0^* - \varepsilon\frac{\partial A_0^*}{\partial Y} + \varepsilon ilA_1^*\right)\\
= &\rho\left(f - i\omega\frac{l}{k}\right)A_0 \times ilA_0^* + \rho\left(f - i\omega\frac{l}{k}\right)A_0 \times\left(-\varepsilon\frac{\partial A_0^*}{\partial Y} + \varepsilon ilA_1^*\right)\\
&+\epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\} \times ilA_0^*\\
= &\rho\left(ilf + \omega\frac{l^2}{k}\right)|A_0|^2 + \rho\varepsilon\left(-fA_0\frac{\partial A_0^*}{\partial Y} + i\omega\frac{l}{k}\frac{\partial A_0^*}{\partial Y}A_0 + ilfA_1^*A_0 + \omega\frac{l^2}{k}A_1^*A_0\right)\\
&+\epsilon\rho\left\{i\frac{l^2\omega}{k^2}A_0^*\frac{\partial A_0}{\partial X} - i\frac{l\omega}{k}A_0^*\frac{\partial A_0}{\partial Y} + i\frac{l^2}{k}A_0^*\frac{\partial A_0}{\partial T} + i\frac{l^2}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0^*A_0 + ilfA_0^*A_1 + \omega\frac{l^2}{k}A_0^*A_1\right\}
\end{aligned}
$$

$$p^*u = \left[\rho\left(f + i\omega\frac{l}{k}\right)A_0^* + \epsilon\rho\left\{\frac{l\omega}{k^2}\frac{\partial A_0^*}{\partial X} - \frac{\omega}{k}\frac{\partial A_0^*}{\partial Y} + \frac{l}{k}\frac{\partial A_0^*}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\frac{l\omega}{k}\right)A_0^* + \left(f + i\omega\frac{l}{k}\right)A_1^*\right\}\right]e^{-i\Theta/\varepsilon}$$
$$\times\left(-ilA_0 - \varepsilon\frac{\partial A_0}{\partial Y} - \varepsilon ilA_1\right)e^{i\Theta/\varepsilon}$$
$$= \left[\rho\left(f + i\omega\frac{l}{k}\right)A_0^* + \epsilon\rho\left\{\frac{l\omega}{k^2}\frac{\partial A_0^*}{\partial X} - \frac{\omega}{k}\frac{\partial A_0^*}{\partial Y} + \frac{l}{k}\frac{\partial A_0^*}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\frac{l\omega}{k}\right)A_0^* + \left(f + i\omega\frac{l}{k}\right)A_1^*\right\}\right]$$
$$\times\left(-ilA_0 - \varepsilon\frac{\partial A_0}{\partial Y} - \varepsilon ilA_1\right)$$
$$= \rho\left(f + i\omega\frac{l}{k}\right)A_0^* \times (-ilA_0) + \rho\left(f + i\omega\frac{l}{k}\right)A_0^* \times \left(-\varepsilon\frac{\partial A_0}{\partial Y} - \varepsilon ilA_1\right)$$
$$+ \epsilon\rho\left\{\frac{l\omega}{k^2}\frac{\partial A_0^*}{\partial X} - \frac{\omega}{k}\frac{\partial A_0^*}{\partial Y} + \frac{l}{k}\frac{\partial A_0^*}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\frac{l\omega}{k}\right)A_0^* + \left(f + i\omega\frac{l}{k}\right)A_1^*\right\} \times (-ilA_0) + O(\epsilon^2)$$
$$= \rho\left(-ilf + \omega\frac{l^2}{k}\right)|A_0|^2 + \rho\varepsilon\left(-fA_0^*\frac{\partial A_0}{\partial Y} - i\omega\frac{l}{k}A_0^*\frac{\partial A_0}{\partial Y} - ilfA_1A_0^* + \omega\frac{l^2}{k}A_0^*A_1\right)$$
$$+ \epsilon\rho\left\{-i\frac{l^2\omega}{k^2}A_0\frac{\partial A_0^*}{\partial X} + i\frac{l\omega}{k}A_0\frac{\partial A_0^*}{\partial Y} - i\frac{l^2}{k}A_0\frac{\partial A_0^*}{\partial T} - i\frac{l^2}{k}\frac{\partial}{\partial X}\left(\frac{l\omega}{k}\right)A_0^*A_0 - ilfA_0A_1^* + \omega\frac{l^2}{k}A_0A_1^*\right\}$$

これから,

$$pu^* + p^*u$$
$$= \rho\left(ilf + \omega\frac{l^2}{k}\right)|A_0|^2 + \rho\varepsilon\left(-fA_0\frac{\partial A_0^*}{\partial Y} + i\omega\frac{l}{k}\frac{\partial A_0^*}{\partial Y}A_0 + ilfA_1^*A_0 + \omega\frac{l^2}{k}A_1^*A_0\right)$$
$$+ \epsilon\rho\left\{i\frac{l^2\omega}{k^2}A_0^*\frac{\partial A_0}{\partial X} - i\frac{l\omega}{k}A_0^*\frac{\partial A_0}{\partial Y} + i\frac{l^2}{k}A_0^*\frac{\partial A_0}{\partial T} + i\frac{l^2}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0^*A_0 + ilfA_0^*A_1 + \omega\frac{l^2}{k}A_0^*A_1\right\}$$
$$+ \rho\left(-ilf + \omega\frac{l^2}{k}\right)|A_0|^2 + \rho\varepsilon\left(-fA_0^*\frac{\partial A_0}{\partial Y} - i\omega\frac{l}{k}A_0^*\frac{\partial A_0}{\partial Y} - ilfA_1A_0^* + \omega\frac{l^2}{k}A_0^*A_1\right)$$
$$+ \epsilon\rho\left\{-i\frac{l^2\omega}{k^2}A_0\frac{\partial A_0^*}{\partial X} + i\frac{l\omega}{k}A_0\frac{\partial A_0^*}{\partial Y} - i\frac{l^2}{k}A_0\frac{\partial A_0^*}{\partial T} - i\frac{l^2}{k}\frac{\partial}{\partial X}\left(\frac{l\omega}{k}\right)A_0^*A_0 - ilfA_0A_1^* + \omega\frac{l^2}{k}A_0A_1^*\right\}$$
$$= 2\rho\omega\frac{l^2}{k}|A_0|^2 + \rho\varepsilon\left\{-f\frac{\partial|A_0|^2}{\partial Y} + i\omega\frac{l}{k}\left(\frac{\partial A_0^*}{\partial Y}A_0 - \frac{\partial A_0}{\partial Y}A_0^*\right) + ilf\left(A_1^*A_0 - A_1A_0^*\right) + \omega\frac{l^2}{k}(A_1^*A_0 + A_0^*A_1)\right\}$$
$$+ \epsilon\rho\left\{i\frac{l^2\omega}{k^2}\left(A_0^*\frac{\partial A_0}{\partial X} - A_0\frac{\partial A_0^*}{\partial X}\right) - i\frac{l\omega}{k}\left(-A_0^*\frac{\partial A_0}{\partial Y} + A_0\frac{\partial A_0^*}{\partial Y}\right)\right.$$
$$\left. + i\frac{l^2}{k}\left(A_0^*\frac{\partial A_0}{\partial T} - A_0\frac{\partial A_0^*}{\partial T}\right) + ilf\left(A_0^*A_1 - A_0A_1^*\right) + \omega\frac{l^2}{k}\left(A_0^*A_1 + A_0A_1^*\right)\right\}$$
$$= 2\rho\omega\frac{l^2}{k}|A_0|^2 + \varepsilon\rho\left\{-f\frac{\partial|A_0|^2}{\partial Y} + 2\omega\frac{l^2}{k}\left(A_0A_1^* + A_0^*A_1\right)\right\}$$
$$+ i\varepsilon\rho\left\{\frac{l^2\omega}{k^2}\left(A_0^*\frac{\partial A_0}{\partial X} - A_0\frac{\partial A_0^*}{\partial X}\right) + \frac{l^2}{k}\left(A_0^*\frac{\partial A_0}{\partial T} - A_0\frac{\partial A_0^*}{\partial T}\right)\right\},$$

$$
\begin{aligned}
pv^* = &\left[\rho\left(f - i\omega\frac{l}{k}\right)A_0\right.\\
&\left.+\epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\}\right]e^{i\Theta/\varepsilon}\\
&\times\left(-ikA_0^* + \varepsilon\frac{\partial A_0^*}{\partial X} - \varepsilon ikA_1^*\right)e^{-i\Theta/\varepsilon}\\
&= \left[\rho\left(f - i\omega\frac{l}{k}\right)A_0 + \epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\}\right]\\
&\times\left(-ikA_0^* + \varepsilon\frac{\partial A_0^*}{\partial X} - \varepsilon ikA_1^*\right)\\
&= \rho\left(fA_0 - i\omega\frac{l}{k}A_0\right)\times(-ikA_0^*) + \rho\varepsilon\left(fA_0 - i\omega\frac{l}{k}A_0\right)\left(\frac{\partial A_0^*}{\partial X} - ikA_1^*\right)\\
&+ \epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0}{\partial Y} + \frac{l}{k}\frac{\partial A_0}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0 + \left(f - i\omega\frac{l}{k}\right)A_1\right\}\times(-ikA_0^*) + O(\epsilon^2)\\
&= \rho\left(-ikf|A_0|^2 - \omega l|A_0|^2\right) + \rho\varepsilon\left(fA_0\frac{\partial A_0^*}{\partial X} - ifkA_0A_1^* - i\omega\frac{l}{k}A_0\frac{\partial A_0^*}{\partial X} - \omega lA_0A_1^*\right)\\
&+ \epsilon\rho\left\{-i\frac{\omega l}{k}A_0^*\frac{\partial A_0}{\partial X} + i\omega A_0^*\frac{\partial A_0}{\partial Y} - ilA_0^*\frac{\partial A_0}{\partial T} - il\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)|A_0|^2 - ikfA_0^*A_1 - \omega lA_0^*A_1\right\},
\end{aligned}
$$

$$
\begin{aligned}
p^*v = &\left[\rho\left(f + i\omega\frac{l}{k}\right)A_0^*\right.\\
&\left.+\epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0^*}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0^*}{\partial Y} + \frac{l}{k}\frac{\partial A_0^*}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0^* + \left(f + i\omega\frac{l}{k}\right)A_1^*\right\}\right]e^{-i\Theta/\varepsilon}\\
&\times\left(ikA_0 + \varepsilon\frac{\partial A_0}{\partial X} + \varepsilon ikA_1\right)e^{i\Theta/\varepsilon}\\
&= \left[\rho\left(f + i\omega\frac{l}{k}\right)A_0^* + \epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0^*}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0^*}{\partial Y} + \frac{l}{k}\frac{\partial A_0^*}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0^* + \left(f + i\omega\frac{l}{k}\right)A_1^*\right\}\right]\\
&\times\left(ikA_0 + \varepsilon\frac{\partial A_0}{\partial X} + \varepsilon ikA_1\right)\\
= &\rho\left(fA_0^* + i\omega\frac{l}{k}A_0^*\right)\times ikA_0 + \rho\varepsilon\left(fA_0^* + i\omega\frac{l}{k}A_0^*\right)\left(\frac{\partial A_0}{\partial X} + ikA_1\right)\\
&+ \epsilon\rho\left\{\omega\frac{l}{k^2}\frac{\partial A_0^*}{\partial X} - \omega\frac{1}{k}\frac{\partial A_0^*}{\partial Y} + \frac{l}{k}\frac{\partial A_0^*}{\partial T} + \frac{l}{k}\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)A_0^* + \left(f + i\omega\frac{l}{k}\right)A_1^*\right\}\times ikA_0 + O(\epsilon^2)\\
= &\rho\left(ikf|A_0|^2 - \omega l|A_0|^2\right) + \rho\varepsilon\left(fA_0^*\frac{\partial A_0}{\partial X} + ikfA_0^*A_1 + i\omega\frac{l}{k}A_0^*\frac{\partial A_0}{\partial X} - \omega lA_0^*A_1\right)\\
&+ \epsilon\rho\left\{i\frac{\omega l}{k}A_0\frac{\partial A_0^*}{\partial X} - i\omega A_0\frac{\partial A_0^*}{\partial Y} + ilA_0\frac{\partial A_0^*}{\partial T} + il\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)|A_0|^2 + ikfA_0A_1^* - \omega lA_0A_1^*\right\}.
\end{aligned}
$$

これから

$$
\begin{aligned}
&pv^* + p^*v \\
&= \rho\left(-ikf|A_0|^2 - \omega l|A_0|^2\right) + \rho\varepsilon\left(fA_0\frac{\partial A_0^*}{\partial X} - ifkA_0A_1^* - i\omega\frac{l}{k}A_0\frac{\partial A_0^*}{\partial X} - \omega lA_0A_1^*\right) \\
&\quad + \epsilon\rho\left\{-i\frac{\omega l}{k}A_0^*\frac{\partial A_0}{\partial X} + i\omega A_0^*\frac{\partial A_0}{\partial Y} - ilA_0^*\frac{\partial A_0}{\partial T} - il\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)|A_0|^2 - ikfA_0^*A_1 - \omega lA_0^*A_1\right\} \\
&\quad + \rho\left(ikf|A_0|^2 - \omega l|A_0|^2\right) + \rho\varepsilon\left(fA_0^*\frac{\partial A_0}{\partial X} + ikfA_0^*A_1 + i\omega\frac{l}{k}A_0^*\frac{\partial A_0}{\partial X} - \omega lA_0^*A_1\right) \\
&\quad + \epsilon\rho\left\{i\frac{\omega l}{k}A_0\frac{\partial A_0^*}{\partial X} - i\omega A_0\frac{\partial A_0^*}{\partial Y} + ilA_0\frac{\partial A_0^*}{\partial T} + il\frac{\partial}{\partial X}\left(\omega\frac{l}{k}\right)|A_0|^2 + ikfA_0A_1^* - \omega lA_0A_1^*\right\} \\
&= -2\rho\omega l|A_0|^2 + \rho\varepsilon\left\{f\frac{\partial|A_0|^2}{\partial X} + ikf\left(A_0^*A_1 - A_0A_1^*\right) + i\frac{\omega l}{k}\left(A_0^*\frac{\partial A_0}{\partial X} - A_0\frac{\partial A_0^*}{\partial X}\right) - \omega l\left(A_0A_1^* + A_0^*A_1\right)\right\} \\
&\quad + \epsilon\rho\left\{i\frac{\omega l}{k}\left(A_0\frac{\partial A_0^*}{\partial X} - A_0^*\frac{\partial A_0}{\partial X}\right) + i\omega\left(A_0^*\frac{\partial A_0}{\partial Y} - A_0\frac{\partial A_0^*}{\partial Y}\right)\right. \\
&\qquad \left. +il\left(A_0\frac{\partial A_0^*}{\partial T} - A_0^*\frac{\partial A_0}{\partial T}\right) + ikf\left(A_0A_1^* - A_0^*A_1\right) - \omega l\left(A_0^*A_1 + A_0A_1^*\right)\right\} \\
&= -2\rho\omega l|A_0|^2 + \rho\varepsilon\left(f\frac{\partial|A_0|^2}{\partial X} - 2\omega l\left(A_0A_1^* + A_0^*A_1\right)\right) \\
&\quad + i\epsilon\rho\left\{\omega\left(A_0^*\frac{\partial A_0}{\partial Y} - A_0\frac{\partial A_0^*}{\partial Y}\right) + l\left(A_0\frac{\partial A_0^*}{\partial T} - A_0^*\frac{\partial A_0}{\partial T}\right)\right\}
\end{aligned}
$$

これまでの結果をまとめて書くと

$$
\begin{aligned}
\overline{pu} &= \frac{1}{2}\rho\omega\frac{l^2}{k}|A_0|^2 + \varepsilon\rho\left\{-\frac{1}{4}f\frac{\partial|A_0|^2}{\partial Y} + \frac{1}{2}\omega\frac{l^2}{k}\left(A_0A_1^* + A_0^*A_1\right)\right\} \\
&\quad + i\frac{1}{2}\varepsilon\rho\left\{\frac{l^2\omega}{k^2}\left(A_0^*\frac{\partial A_0}{\partial X} - A_0\frac{\partial A_0^*}{\partial X}\right) + \frac{l^2}{k}\left(A_0^*\frac{\partial A_0}{\partial T} - A_0\frac{\partial A_0^*}{\partial T}\right)\right\} \\
\overline{pv} &= -\frac{1}{2}\rho\omega l|A_0|^2 + \rho\varepsilon\left(\frac{1}{4}f\frac{\partial|A_0|^2}{\partial X} - \frac{1}{2}\omega l\left(A_0A_1^* + A_0^*A_1\right)\right) \\
&\quad + i\frac{1}{2}\epsilon\rho\left\{\omega\left(A_0^*\frac{\partial A_0}{\partial Y} - A_0\frac{\partial A_0^*}{\partial Y}\right) + l\left(A_0\frac{\partial A_0^*}{\partial T} - A_0^*\frac{\partial A_0}{\partial T}\right)\right\}
\end{aligned}
$$

となる.

i がついた項は無視して分散関係, 群速度を用いて書き直す. $O(\epsilon^0)$ の項は

$$
\begin{aligned}
\frac{1}{2}\rho\omega\frac{l^2}{k}|A_0|^2 &= \frac{1}{2}\rho\left(\frac{-\beta k}{k^2+l^2}\right)\frac{l^2}{k}|A_0|^2 = -\frac{1}{2}\rho\beta\frac{l^2}{k^2+l^2}|A_0|^2 \\
&= -\frac{1}{4}\rho\beta\frac{2l^2}{k^2+l^2}|A_0|^2 = -\frac{1}{4}\rho\beta\frac{l^2+l^2+k^2-k^2}{k^2+l^2}|A_0|^2 \\
&= -\frac{1}{4}\rho\beta|A_0|^2 - \frac{1}{4}\rho\beta\left(\frac{l^2-k^2}{k^2+l^2}\right)|A_0|^2 \\
-\frac{1}{2}\rho\omega l|A_0|^2 &= -\frac{1}{2}\rho\left(\frac{-\beta k}{k^2+l^2}\right)l|A_0|^2 = \frac{1}{2}\rho\beta\frac{kl}{k^2+l^2}|A_0|^2
\end{aligned}
$$

ここで,

$$
\begin{aligned}
C_{gx}\overline{E} &= \frac{\beta(k^2-l^2)}{(k^2+l^2)^2}\times\frac{\rho}{4}(k^2+l^2)|A_0|^2 = \frac{\rho}{4}\frac{\beta(k^2-l^2)}{k^2+l^2}|A_0|^2,\\
C_{gy}\overline{E} &= \frac{2\beta kl}{(k^2+l^2)^2}\times\frac{\rho}{4}(k^2+l^2)|A_0|^2 = \frac{\rho}{2}\frac{\beta kl}{k^2+l^2}|A_0|^2
\end{aligned}
$$

を使うと

$$
\begin{aligned}
\overline{pu} &= -\frac{1}{2}\rho\beta\frac{l^2}{k^2+l^2}|A_0|^2 + \varepsilon\rho\left\{\frac{1}{2}\omega\frac{l^2}{k}(A_0A_1^* + A_0^*A_1) - \frac{1}{4}f\frac{\partial}{\partial Y}|A_0|^2\right\}\\
&= \frac{1}{4}\rho\beta\frac{k^2-l^2}{k^2+l^2}|A_0|^2 - \frac{1}{4}\rho\beta|A_0|^2 + \varepsilon\rho\left\{\frac{1}{2}\omega\frac{l^2}{k}(A_0A_1^* + A_0^*A_1) - \frac{1}{4}f\frac{\partial}{\partial Y}|A_0|^2\right\}\\
&= c_{gx}\bar{E} - \frac{1}{4}\rho\beta|A_0|^2 + \varepsilon\rho\left\{\frac{1}{2}\omega\frac{l^2}{k}(A_0A_1^* + A_0^*A_1) - \frac{1}{4}f\frac{\partial}{\partial Y}(|A_0|^2)\right\}.\\
\overline{pv} &= \frac{1}{2}\rho\beta\frac{kl}{k^2+l^2}|A_0|^2 + \varepsilon\rho\left\{-\frac{1}{2}\omega l(A_0A_1^* + A_0^*A_1) + \frac{1}{4}f\frac{\partial}{\partial X}|A_0|^2\right\}\\
&= c_{gy}\bar{E} + \varepsilon\rho\left\{-\frac{1}{2}\omega l(A_0A_1^* + A_0^*A_1) + \frac{1}{4}f\frac{\partial}{\partial X}|A_0|^2\right\}.
\end{aligned}
$$

かくして, $O(\varepsilon^0)$ では

$$
\overline{p\boldsymbol{v}} = \boldsymbol{c}_g\bar{E} - \frac{1}{4}\rho\beta|A_0|^2\hat{\boldsymbol{x}} \tag{2.52}
$$

となる. $\overline{p\boldsymbol{v}}$ は 群速度の方向からずれてしまうことがわかる.

しかしながら, フラックスの収束発散は $\partial f/\partial Y = \beta/\varepsilon$ であり, 従って $O(\varepsilon^1)$ からの寄与を考慮すれば

$$
\begin{aligned}
\frac{\partial}{\partial \boldsymbol{X}}\overline{p\boldsymbol{v}} &= \frac{\partial}{\partial \boldsymbol{X}}(\boldsymbol{c}_g\bar{E}) + \frac{\partial}{\partial X}\left(-\frac{1}{4}\rho\beta|A_0|^2\right) + \frac{\partial}{\partial Y}\left(\varepsilon\frac{1}{4}f\frac{\partial}{\partial X}|A_0|^2\right)\\
&= \frac{\partial}{\partial \boldsymbol{X}}(\boldsymbol{c}_g\bar{E}) - \frac{1}{4}\rho\beta\frac{\partial}{\partial X}|A_0|^2 + \frac{1}{4}\rho\beta\frac{\partial}{\partial X}|A_0|^2\\
&= \frac{\partial}{\partial \boldsymbol{X}}(\boldsymbol{c}_g\bar{E})
\end{aligned} \tag{2.53}
$$

$\overline{p\boldsymbol{v}}$ と $\boldsymbol{c}_g\bar{E}$ とのずれは非発散であり, 従ってエネルギーの正味の変化にはかかわりのないものであることがわかる.

実際に, $p\boldsymbol{v}$ の発散をとってみると

$$
\begin{aligned}
\frac{\partial}{\partial \boldsymbol{X}}\overline{p\boldsymbol{v}} = & \frac{\partial}{\partial X}\left[c_{gx}\bar{E} - \frac{1}{4}\rho\beta|A_0|^2 + \varepsilon\rho\left\{\frac{1}{2}\omega\frac{l^2}{k}(A_0A_1^* + A_0^*A_1) - \frac{1}{4}f\frac{\partial}{\partial Y}(|A_0|^2)\right\}\right. \\
& \left. +i\frac{1}{2}\varepsilon\rho\left\{\frac{l^2\omega}{k^2}\left(A_0^*\frac{\partial A_0}{\partial X} - A_0\frac{\partial A_0^*}{\partial X}\right) + \frac{l^2}{k}\left(A_0^*\frac{\partial A_0}{\partial T} - A_0\frac{\partial A_0^*}{\partial T}\right)\right\}\right] \\
& + \frac{\partial}{\partial Y}\left[c_{gy}\bar{E} + \varepsilon\rho\left\{-\frac{1}{2}\omega l(A_0A_1^* + A_0^*A_1) + \frac{1}{4}f\frac{\partial}{\partial X}|A_0|^2\right\}\right. \\
& \left. +i\frac{1}{2}\epsilon\rho\left\{\omega\left(A_0^*\frac{\partial A_0}{\partial Y} - A_0\frac{\partial A_0^*}{\partial Y}\right) + l\left(A_0\frac{\partial A_0^*}{\partial T} - A_0^*\frac{\partial A_0}{\partial T}\right)\right\}\right] \\
= & \frac{\partial}{\partial \boldsymbol{X}}\left(\boldsymbol{c}_g\bar{E}\right) + \frac{\partial}{\partial X}\left[-\frac{1}{4}\rho\beta|A_0|^2\right] + \frac{\partial}{\partial Y}\left[\varepsilon\rho\left\{\frac{1}{4}f\frac{\partial}{\partial X}|A_0|^2\right\}\right] + O(\epsilon) \\
= & \frac{\partial}{\partial \boldsymbol{X}}\left(\boldsymbol{c}_g\bar{E}\right) - \frac{1}{4}\rho\beta\frac{\partial|A_0|^2}{\partial X} + \frac{1}{4}\varepsilon\rho\frac{\partial f}{\partial Y}\frac{\partial|A_0|^2}{\partial X} + \frac{1}{4}\varepsilon\rho f\frac{\partial}{\partial Y}\frac{\partial|A_0|^2}{\partial X} + O(\epsilon) \\
= & \frac{\partial}{\partial \boldsymbol{X}}\left(\boldsymbol{c}_g\bar{E}\right) - \frac{1}{4}\rho\beta\frac{\partial|A_0|^2}{\partial X} + \frac{1}{4}\varepsilon\rho\frac{\beta}{\epsilon}\frac{\partial|A_0|^2}{\partial X} + \frac{1}{4}\varepsilon\rho f\frac{\partial}{\partial Y}\frac{\partial|A_0|^2}{\partial X} + O(\epsilon) \\
= & \frac{\partial}{\partial \boldsymbol{X}}\left(\boldsymbol{c}_g\bar{E}\right) + O(\epsilon)
\end{aligned}
$$

と確認できる.

第3章 ロスビー波の例

3.1 ロスビー波の例

地球大気に生じたロスビー波の例を図に示す.

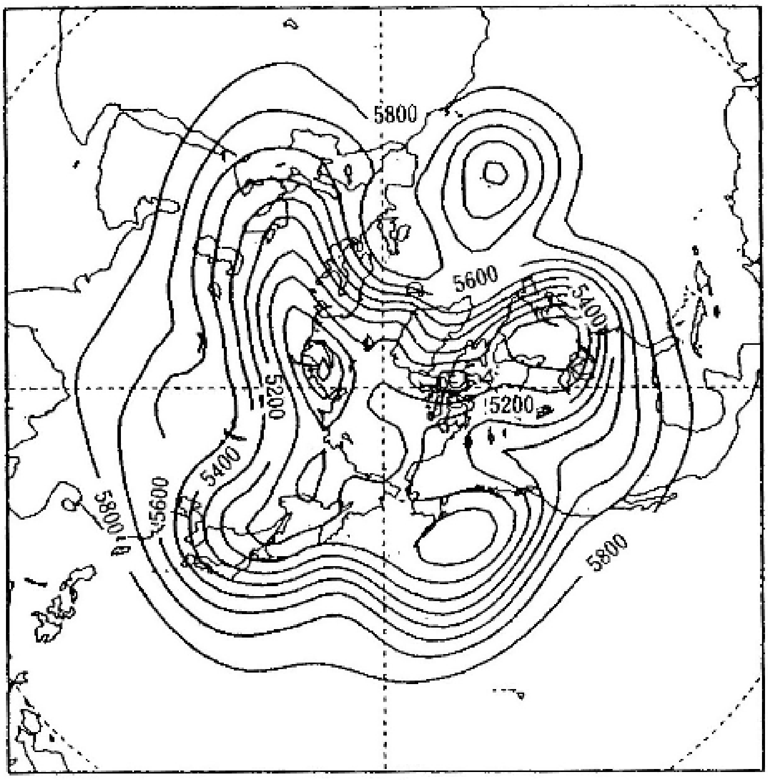

図6.

北半球 500 mb 等圧面高度分布図 (1987 年 1 月 24 日)
単位は m.

図 3.1: 偏西風の蛇行がロスビー波を表している.

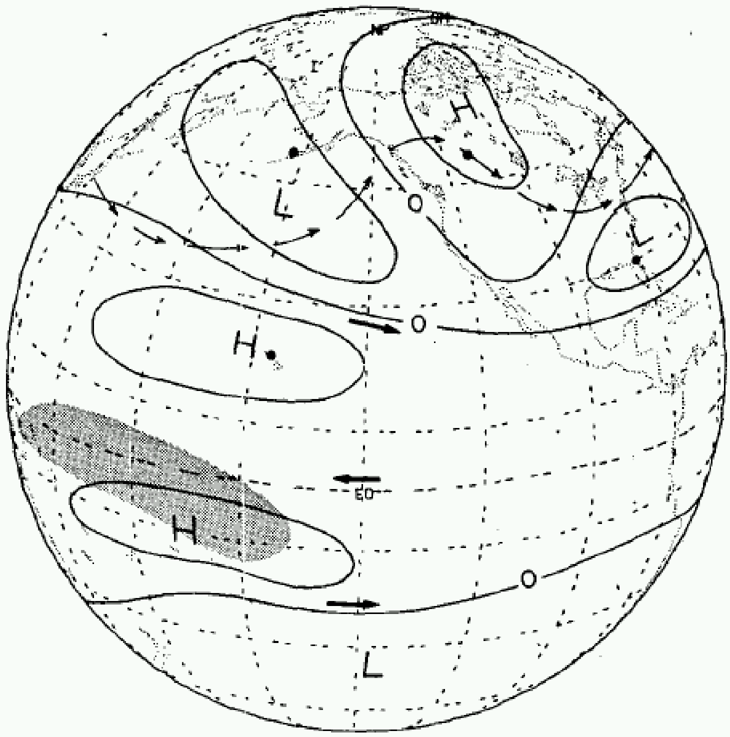

FIG. 11. Schematic illustration of the hypothesized global pattern of middle and upper tropospheric geopotential height anomalies (solid lines) during a Northern Hemisphere winter which falls within an episode of warm sea surface temperatures in the equatorial Pacific. The arrows in darker type reflect the strengthening of the subtropical jets in both hemispheres along with stronger easterlies near the equator during warm episodes. The arrows in lighter type depict a mid-tropospheric streamline as distorted by the anomaly pattern, with pronounced "troughing" over the central Pacific and "ridging" over western Canada. Shading indicates regions of enhanced cirriform cloudiness and rainfall. For further details see Section 7. The locations of the stations used in Table 4 are indicated by dots.

図 3.2: Horel and Wallace (1981) の Fig.11. エルニーニョ現象に伴う海面水温偏差によりロスビー波の波列が生じると議論された.

参考文献

Horel, J. D., and Wallace, J. M., 1981: Planetary-Scale Atmospheric Phenomena Associated with the Southern Oscillation. *Mon. Wea. Rev.*, **109**, 813-829.

http://www2.nict.go.jp/y/y222/SMILES/MACS/9/note/hadou.html